WARING'S PROBLEM WITH DIGITAL RESTRICTIONS IN $\mathbb{F}_q[X]$

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ABSTRACT. We present a generalization of a result due to Thuswaldner and Tichy to the ring of polynomials over a finite fields. In particular we want to show that every polynomial of sufficiently large degree can be represented as sum of k-th powers, where the bases evaluated on additive functions meet certain congruence restrictions.

1. INTRODUCTION

In the present paper we want to show a generalization of a result due to Thuswaldner and Tichy [13] to the ring of polynomials over a finite field. In a recent paper they could prove that for fixed positive integers j and m every sufficiently large positive integer N has a representation of the form

$$N = n_1^k + \cdots + n_s^k \quad (s_q(n_i) \equiv j \bmod m, 1 \leq i \leq s),$$

where s_q is the q-ary sum of digits function. This result has been further generalized to different congruences for each summand by Pfeifer and Thuswaldner [12] and to arbitrary q-additive functions by Wagner [14].

In order to carry these results over to the ring of polynomials over a finite field $\mathcal{R} := \mathbb{F}_q[X]$ we start with a definition of a number system in this ring. To this matter we fix a polynomial $Q \in \mathcal{R}$ of positive degree d. It is easy to see that each $A \in \mathcal{R}$ admits a unique and finite *Q-ary digital expansion* of the form

$$A = \sum_{i \geq 0} D_i Q^i \tag{1.1}$$

with $D_i \in \mathcal{R}$ and $\deg D_i < \deg Q$. We call a function $f : \mathcal{R} \to G$, where G is an Abelian group, *strongly Q-additive* if $f(AQ + B) = f(A) + f(B)$. Thus, if we represent an element $A \in \mathcal{R}$ by its Q-ary digital expansion (1.1), we may write

$$f(A) = \sum_{i \geq 0} f(D_i).$$

One simple example is the Q-ary *sum of digits function*, which is defined by

$$s_Q(A) := \sum_{i \geq 0} D_i.$$

As the results of Thuswaldner and Tichy are based on Waring's Problem we take a closer look at generalizations of this problem to the ring $\mathcal{R}$. Let $\mathcal{A} \subset \mathcal{R}$ and s be a positive integer. We call $\mathcal{A}$ a *basis* of $\mathcal{R}$ of *order* s if for every $N \in \mathcal{R}$ there is at least one representation of the form

$$N = P_1 + \cdots + P_s \quad \text{with } P_1, \ldots, P_s \in \mathcal{A}.$$

We call $\mathcal{A}$ an *asymptotic basis* if this is true for all N of sufficiently large degree.

Now the generalization of Waring's Problem is the question whether $\mathcal{A} := \{A^k : A \in \mathcal{R}\}$ is an asymptotic basis of $\mathcal{R}$. This question was positively answered by Paley [11]. As in the case of integers one is also interested in an asymptotic for the number of solutions as it is provided by the circle method. The problem with $\mathcal{R}$ is, that if one has one representation, then one gets infinitely

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many by cancellation in the higher degree terms. In order to prevent this side effect one bounds the degree of the basis. Therefore we are interested in the number of solutions of

$$N = P_1 + \cdots + P_s \quad \text{with } P_1, \ldots, P_s \in \mathcal{A} \quad \text{and} \quad \deg P_i \leq \deg N \quad (i = 1, \ldots, s). \tag{1.2}$$

For $\mathcal{A} := \{A^k : A \in \mathcal{R}\}$ this was considered independently by Car [1] and Kubota [9].

For $\mathcal{A} := \{A : A \in \mathcal{R} \text{ and } A \text{ irreducible}\}$, which corresponds to Goldbach's Problem, Hayes [8] considered the number of solutions.

Another variant is the question if it is possible to represent every polynomial N as the sum of two irreducible polynomials and a kth power, *i.e.*,

$$N = P_1 + P_2 + A^k, \quad P_1, P_2 \text{ irreducible}, A \in \mathcal{R}.$$

This problem was considered by Car in [2].

Finally there is a further variant which deals with the problem that even with the statement in (1.2) we count some solutions twice. Therefore we could refine the problem a little bit further and consider

$$N = P_1^k + \cdots + P_s^k \quad \text{with } \deg N = nk, \deg P_i = n \quad (i = 1, \ldots, s). \tag{1.3}$$

This is called *strict* Waring's Problem and was considered by Webb [15].

In the present paper we focus on the number of solutions of (1.2). We are interested in sets with digital restrictions that are asymptotic bases for $\mathcal{R}$. Throughout the rest of the paper let f_i denote a strongly Q_i-additive function where $Q_i \in \mathcal{R}$ are pairwise coprime polynomials and $d_i := \deg Q_i$. Furthermore let $M_i \in \mathcal{R}$ and $m_i = \deg M_i$ for $i = 1, \ldots, r$.

We denote by

$$\mathcal{P}_m = \{A \in \mathcal{R} : \deg A < m\}$$

the set of polynomials of degree less than m. Then we want to consider the set

$$\mathcal{C}_m(\mathbf{f}, \mathbf{J}, \mathbf{M}) = \mathcal{C}_m(\mathbf{J}) := \{A \in \mathcal{P}_m : f_1(A) \equiv J_1 \bmod M_1, \ldots, f_r(A) \equiv J_r \bmod M_r\}.$$

Moreover, let

$$\mathcal{C}(\mathbf{f}, \mathbf{J}, \mathbf{M}) = \mathcal{C}(\mathbf{J}) := \bigcup_{m \geq 1} \mathcal{C}_m(\mathbf{J}). \tag{1.4}$$

Properties of similar sets have been investigated by Drmota and Gutenbrunner [5]. For estimates of Weyl sums over these sets one may consider Madritsch and Thuswaldner [10].

We call $r(N, n, s, k, q)$ the number of solutions of (1.2). Then Kubota [9] could prove the following result.

Proposition 1.1 ([9, Theorem 30])**.** *If $0 < \varepsilon < 1$, $\deg N < (k-1+\varepsilon)n$, $s \geq 2^k+1$, $3 \leq k < \operatorname{char} F_q$, then there exists $\delta > 0$ such that*

$$r(N, n, s, k, q) = \mathfrak{S}(N, s, k, q) q^{(s-k)n} + \mathcal{O}\left(q^{(s-k-\delta)n}\right),$$

where

$$1 \ll \mathfrak{S}(N, s, k, q) \ll 1.$$

The assumption that $k \geq 3$ is motivated by the fact that if $k = 2$ then there are no minor arcs. In this case the number of solutions has been considered by Carlitz [3, 4].

In this paper we want to count the number of solutions of (1.2) with $\mathcal{A} = \mathcal{C}(\mathbf{f}, \mathbf{J}, \mathbf{M})$. Then our theorem for the polynomial Waring reads as follows.

Theorem 1.2. *Let $Q_1, \ldots, Q_r \in \mathcal{R}$ be relatively prime and for $i \in \{1, \ldots, r\}$ let f_i be a Q_i-additive function. Choose $M_1, \ldots, M_r, J_1, \ldots, J_r \in \mathcal{R}$ and set $m_i := \deg M_i$ $(i = 1, \ldots, r)$. Suppose that for every $\mathbf{0} \neq \mathbf{R} \in \mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_r}$ there exists an $A \in \mathcal{R}$ such that*

$$g_0(A) = E\left(\sum_{i=1}^{r} \frac{R_i}{M_i} f_i(A)\right) \neq 1,$$

where E is the character defined in (2.2)*. Let $N \in \mathcal{R}$. If $3 \le k < p = \operatorname{char} \mathbb{F}_q$ and $m = \lceil \deg N/k \rceil + 1$, then for $s \ge k2^k$ and for every N with sufficiently large degree we always get a solution for*

$$N = P_1^k + \cdots + P_s^k, \quad \text{with} \quad P_i \in \mathcal{C}_m(\mathbf{f}, \mathbf{J}, \mathbf{M}) \text{ for } i = 1, \ldots, s. \tag{1.5}$$

Moreover, let $R(N, n, s, k, q, \mathbf{f}, \mathbf{J}, \mathbf{M})$ denote the number of solutions of (1.5)*, then we get that*

$$R(N, n, s, k, q, \mathbf{f}, \mathbf{J}, \mathbf{M}) = q^{-s \sum_{j=1}^{r} m_j} r(N, n, s, k, q) + \mathcal{O}\left(q^{n(s-k)-n/k}\right).$$

where $r(N, n, s, k, q)$ is as in Proposition 1.1.

We split the proof into two major parts. First in Section 2 we collect some tools that we will need in order to prove Theorem 1.2. Then in Section 3 we give the proof.

Remark 1.3. We can further generalize Theorem 1.2 such that every P_i for $i = 1, \ldots, s$ has its own congruence set $\mathcal{C}_{n,t}(\mathbf{f}_t, \mathbf{J}_t, \mathbf{M}_t)$. This goes down the same lines but with tedious index notation.

Remark 1.4. In the same manner as in the following sections one can also prove a similar result for the strict Problem of Waring. Therefore it suffices to replace the Theorem of Kubota [9] by the one of Webb [15] and proceed as in the proof of Webb.

2. Preliminaries and Lemmata

We start by stating the definitions and settings for the proof of Theorem 1.2. All these objects are standard in this field (see for instance [2, 9]) and we recall their definition briefly.

We set $\mathcal{K} := \mathbb{F}_q(X)$ for the field of rational polynomials over $\mathbb{F}_q$. Moreover, vectors will be written in boldface, *i.e.*, we will write for instance $\mathbf{D} := (D_1, \ldots, D_\ell)$ where ℓ is an integer.

With $\mathcal{R}$ and $\mathcal{K}$ we have the analogues for the ring of "integers" and the field of "rationals", respectively. To get an equivalent for the "reals" we define a valuation ν (the inverse degree valuation or valuation at infinity) as follows. Let $A, B \in \mathcal{R}$, then

$$\nu(A/B) := \deg B - \deg A \tag{2.1}$$

and $\nu(0) := \infty$. With help of this valuation we can complete $\mathcal{K}$ to the field $\mathcal{K}_\infty := \mathbb{F}_q((X^{-1}))$ of formal Laurent series. Then we get

$$\nu\left(\sum_{i=-\infty}^{+\infty} a_i X^i\right) = -\sup\{i \in \mathbb{Z} : a_i \neq 0\}.$$

Thus for $A \in \mathcal{R}$ we have $\nu(A) = -\deg A$.

For convenience if not stated otherwise we will always denote a polynomial in $\mathcal{R}$ by a big Latin letter and a formal Laurent series in $\mathcal{K}_\infty$ by a small Greek letter.

We equip the group $(\mathcal{K}_\infty, +)$ with a Haar measure and normalize it. To this matter we denote by $\mathcal{U}(\ell) := \{A \in \mathcal{K}_\infty : \nu(A) < \ell\}$. We call $\mathcal{U}_\infty := \mathcal{U}(0)$ the *unit interval.* We normalize the Haar measure on $\mathcal{K}_\infty$ such that

$$\int_{\alpha \in \mathcal{U}_\infty} 1 \cdot \mathrm{d}\alpha = 1.$$

Thus we get by the invariance of the Haar measure under addition that for all $\beta \in \mathcal{K}_\infty$

$$\int_{\nu(\alpha-\beta)<n} 1 \cdot \mathrm{d}\alpha = q^{-n}.$$

The next ingredient for the Weyl Sums are additive characters. Let $\alpha \in \mathcal{K}_\infty$, $\alpha = \sum_{i=-\infty}^{\nu(\alpha)} a_i X^i$. Then by $\operatorname{Res} \alpha := a_{-1}$ we denote the residue of an element α. In a finite field $\mathbb{F}_q$ of characteristic $\operatorname{char} \mathbb{F}_q = p$ we define the additive character E by

$$E(\alpha) := \exp\left(2\pi i \operatorname{tr}(\operatorname{Res} \alpha)/p\right), \tag{2.2}$$

where $\operatorname{tr} : \mathbb{F}_q \to \mathbb{F}_p$ denotes the usual trace of an element of $\mathbb{F}_q$ in $\mathbb{F}_p$.

This character has the following basic properties which mainly correspond to well-known properties of the character $\exp(2\pi i x)$.

Lemma 2.1 ([9, Lemma 1])**.**

(1) *If* $\nu(\alpha-\beta)>1$ *then* $E(\alpha)=E(\beta)$.
(2) $E:K_\infty\to\mathbb{C}$ *is continuous.*
(3) E *is not identically* 1.
(4) $E(\alpha+\beta)=E(\alpha)E(\beta)$.
(5) $E(A)=1$ *for every* $A\in\mathbb{F}_q[X]$.
(6) *For* $n\in\mathbb{Z}$ *and* $N\in\mathcal{R}$ *we have*

$$\int_{\alpha\in\mathcal{U}(n)}E(\alpha N)\mathrm{d}\alpha=\begin{cases}q^{-n} & \text{if } \deg N<n,\\ 0 & \text{otherwise.}\end{cases}$$

(7) *For* $N,Q\in\mathcal{R}$ *we have*

$$\sum_{\deg A<\deg Q}E\left(\frac{A}{Q}N\right)=\begin{cases}q^{\deg Q} & \text{if } Q|N,\\ 0 & \text{otherwise.}\end{cases}$$

Now we need two further tools. The first one is the corresponding version of Weyl's inequality. Therefore we define the difference operator Δ_ℓ ($\ell\geq 0$) for a function φ recursively by

$$\Delta_0(\varphi(A)):=\varphi(A),$$
$$\Delta_{\ell+1}(\varphi(A);D_1,\dots,D_{\ell+1}):=\Delta_\ell(\varphi(A+D_{\ell+1});D_1,\dots,D_\ell)-\Delta_\ell(\varphi(A);D_1,\dots,D_\ell).$$

Lemma 2.2 ([10, Theorem 2.2])**.** *Let* $Q_1,\dots,Q_r\in\mathcal{R}$ *be relatively prime and for* $i\in\{1,\dots,r\}$ *let* f_i *be a* Q_i*-additive function. Choose* $M_1,\dots,M_r\in\mathcal{R}$*, set* $m_i:=\deg M_i$*, and fix* $\mathbf{R}\in\mathcal{P}_{m_1}\times\cdots\times\mathcal{P}_{m_r}$*. If there exists* $\mathbf{H}\in\mathcal{R}^k$ *and* $A\in\mathcal{R}$ *such that*

$$E\left(\sum_{i=1}^r\frac{R_i}{M_i}\Delta_k(f_i(A);\mathbf{H})\right)\neq 1,$$

then

$$\sum_{A\in\mathcal{P}_n}E\left(\alpha A^k+\sum_{i=1}^r\frac{R_i}{M_i}f_i(A)\right)\ll q^{n\left(1-2^{-k-1}\gamma\right)},$$

where

$$\gamma=2+\frac{k}{2}+\frac{1-\left|\Phi_{i,k}(\mathbf{H};d_i)\right|^2}{d_iq^{d_i}}$$

with some constant $|\Phi_{i,k}(\mathbf{H};d_i)|\in(0,1)$.

The second one is an analogue to Hua's Lemma for $\mathcal{R}$.

Lemma 2.3 (*cf.* Theorem 8.13 in [6])**.** *Let* $F(Y)$ *be a polynomial over* $\mathcal{R}$ *and let* ℓ *be an integer such that* $\Delta_\ell(F(Y);Y_1,\dots,Y_\ell)\in\mathcal{R}[Y,Y_1,\dots,Y_\ell]$ *and*

$$\Delta_\ell(F(Y);Y_1,\dots,Y_\ell)\neq 0.$$

Then, for every $\varepsilon>0$,

$$\int_{\alpha\in U_\infty}\left|\sum_{P\in\mathcal{P}_n}E(\alpha F(P))\right|^{2^\ell}\mathrm{d}\alpha\ll q^{n(2^\ell-\ell+\varepsilon)}.$$

3. Proof of Theorem 1.2

The proof of Theorem 1.2 makes use of the circle method and we mainly follow Webb [15] and Thuswaldner and Tichy [13]. We adopt their method and denote by $R(N):=R(N,n,s,k,\mathbf{J},\mathbf{M},q)$ the number of solutions of the equation

$$N=P_1^k+\cdots+P_s^k,\quad (P_i\in\mathcal{C}_n(\mathbf{J})\text{ for }1\leq i\leq s).$$

The Weyl sum under consideration is defined as

$$S_n(\alpha) := \sum_{P \in \mathcal{C}_n(\mathbf{J})} E(\alpha P^k).$$

Hence, by Lemma 2.1(6) we get

$$R(N) = \int_{\alpha \in U_\infty} S_n(\alpha) \cdots S_n(\alpha) E(-N\alpha) \mathrm{d}\alpha. \tag{3.1}$$

In order to change the range of summation from $\mathcal{C}_n(\mathbf{J})$ to $\mathcal{P}_n$ we adopt an idea of Gelfond [7]. Thus we may rewrite $S_n(\alpha)$ as

$$S_n(\alpha) = q^{-\sum_{j=1}^r m_j} \sum_{\mathbf{R} \in \mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_r}} \sum_{P \in \mathcal{P}_n} E\left(\sum_{i=1}^r \frac{R_i}{M_i}(f_i(P) - J_i)\right) E(\alpha P^k).$$

Plugging this into (3.1) yields

$$\begin{aligned} R(N) = q^{-s\sum_{j=1}^r m_j} & \int_{\alpha \in U_\infty} \sum_{P_1 \in \mathcal{P}_n} \cdots \sum_{P_s \in \mathcal{P}_n} \sum_{\mathbf{R} \in \mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_r}} \\ & \times E\left(\sum_{i=1}^r \frac{R_i}{M_i}(f_i(P_1) - J_i)\right) \cdots E\left(\sum_{i=1}^r \frac{R_i}{M_i}(f_i(P_s) - J_i)\right) \\ & \times E(\alpha(P_1^k + \cdots + P_s^k - N)) \mathrm{d}\alpha. \end{aligned}$$

We split the integral up into two parts according to $\mathbf{R}$ and get

$$R(N) = q^{-s\sum_{j=1}^r m_j}(I_1 + I_2), \tag{3.2}$$

where

$$\begin{aligned} I_1 &= \int_{\alpha \in U_\infty} \sum_{P_1 \in \mathcal{P}_n} \cdots \sum_{P_s \in \mathcal{P}_n} E(\alpha(P_1^k + \cdots + P_s^k - N)) \mathrm{d}\alpha, \\ I_2 &= \int_{\alpha \in U_\infty} \sum_{P_1 \in \mathcal{P}_n} \cdots \sum_{P_s \in \mathcal{P}_n} \sum_{\mathbf{0} \neq \mathbf{R} \in \mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_r}} \\ & \times E\left(\sum_{i=1}^r \frac{R_i}{M_i}(f_i(P_1) - J_i)\right) \cdots E\left(\sum_{i=1}^r \frac{R_i}{M_i}(f_i(P_s) - J_i)\right) \\ & \times E(\alpha(P_1^k + \cdots + P_s^k - N)) \mathrm{d}\alpha. \end{aligned}$$

Noting Lemma 2.1(6) we get that

$$I_1 = r(N, n, s, k, q) \tag{3.3}$$

and we may apply Proposition 1.1.

As we will see, I_2 will contribute to the error term. From now on we assume that $\mathbf{R} \neq \mathbf{0}$. Then we get

$$I_2 = \sum_{R_1 \in \mathcal{P}_m} \cdots \sum_{R_s \in \mathcal{P}_m} I_{\mathbf{R}}$$

where

$$I_{\mathbf{R}} := \int_{\alpha \in U_\infty} \prod_{t=1}^s S_{n,t}(\alpha) E(-\alpha N) \mathrm{d}\alpha,$$

$$S_{n,t}(\alpha) := \sum_{P \in \mathcal{P}_n} E\left(\alpha P^k + \sum_{i=1}^r \frac{R_i}{M_i}(f_i(P) - J_i)\right).$$

To estimate $I_{\mathbf{R}}$ we split the integral up into two parts according to $s > 2^k$ and get

$$|I_{\mathbf{R}}| \leq \sup_{\alpha, t}(|S_{n,t}(\alpha)|^{s-2^k}) \max_t \left(\int_{\alpha \in U_\infty} |S_{n,t}(\alpha)|^{2^k} \mathrm{d}\alpha\right).$$

For the supremum we apply Lemma 2.2. The integral is estimated by the same trick as by Thuswaldner and Tichy [13]. Noting that

$$\int_{\alpha\in U_\infty} |S_{n,i}(\alpha)|^{2^k}\,\mathrm{d}\alpha = \sum_{\mathbf{P}\in\mathcal{P}_n^{2^k}} E\left(\sum_{i=1}^{r}\frac{R_i}{M_i}\sum_{t=1}^{2^{k-1}} f_i(P_t)-f_i(P_{t+2^{k-1}})\right),$$

where the sum is over all $\mathbf{P}\in\mathcal{P}_n^{2^k}$ such that

$$P_1^k+\cdots+P_{2^{k-1}}^k = P_{2^{k-1}+1}^k+\cdots+P_{2^k}^k.$$

We estimate the sum with the number of solutions of this equation trivially and get

$$\int_{\alpha\in U_\infty} |S_{n,t}(\alpha)|^{2^k}\,\mathrm{d}\alpha \ll \int_{\alpha\in U_\infty}\left|\sum_{P\in\mathcal{P}_n} E(\alpha P^k)\right|^{2^k}\mathrm{d}\alpha. \tag{3.4}$$

For the last integral we apply Hua's Lemma (Lemma 2.3) to obtain

$$\int_{\alpha\in U_\infty} |S_{n,i}(\alpha)|^{2^k}\,\mathrm{d}\alpha \ll q^{n(2^k-k+\varepsilon)}.$$

Together with Lemma 2.2 for the supremum this yields for I_2

$$I_2 \ll q^{n(1-2^{-k-1}-\gamma)(s-2^k)}\,q^{n(2^k-k+\varepsilon)} \ll q^{n(s-k)-n/k} \tag{3.5}$$

where γ is as in Lemma 2.2.

Plugging (3.3) and (3.5) into (3.2) we get

$$R(N,n,s,k,q,\mathbf{f},\mathbf{J},\mathbf{M}) = q^{-s\sum_{j=1}^r m_j} r(N,n,s,k,q)+\mathcal{O}\left(q^{n(s-k)-n/k}\right),$$

which proves the theorem.

References

[1] M. Car, *Le problème de Waring pour l'anneau des polynômes sur un corps fini*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A141–A144.

[2] ———, *Sommes de carrés de polynômes irréductibles dans* $\mathbf{F}_q[X]$, Acta Arith. **44** (1984), no. 4, 307–321.

[3] L. Carlitz, *Representations of arithmetic functions in* $gf[p^n,x]$. *i.*, Duke Math. J. **14** (1947), 1121–1137.

[4] ———, *Representations of arithmetic functions in* $GF[p^n,x]$. *II.*, Duke Math. J. **15** (1948), 795–801.

[5] M. Drmota and G. Gutenbrunner, *The joint distribution of Q-additive functions on polynomials over finite fields*, J. Théor. Nombres Bordeaux **17** (2005), no. 1, 125–150.

[6] G. W. Effinger and D. R. Hayes, *Additive number theory of polynomials over a finite field*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1991, , Oxford Science Publications.

[7] A. O. Gel′fond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1967/1968), 259–265.

[8] D. R. Hayes, *The expression of a polynomial as a sum of three irreducibles*, Acta Arith. **11** (1966), 461–488.

[9] R. M. Kubota, *Waring's problem for* $\mathbf{F}_q[x]$, Dissertationes Math. (Rozprawy Mat.) **117** (1974), 60.

[10] M. G. Madritsch and J. M. Thuswaldner, *Weyl sums in* $\mathbb{F}_q[x]$ *with digital restrictions*, Finite Fields Appl. **14** (2008), no. 4, 877–896.

[11] R. E. A. C. Paley, *Theorems on polynomials in a Galois field.*, Quarterly Journ. (Oxford Series) **4** (1934), 52–63 (English).

[12] O. Pfeiffer and J. M. Thuswaldner, *Waring's problem restricted by a system of sum of digits congruences*, Quaest. Math. **30** (2007), no. 4, 513–523.

[13] J. M. Thuswaldner and R. F. Tichy, *Waring's problem with digital restrictions*, Israel J. Math. **149** (2005), 317–344, Probability in mathematics.

[14] S. Wagner, *Waring's problem with restrictions on q-additive functions*, Math. Slovaca. (2008), to appear.

[15] W. A. Webb, *Waring's problem in* GF[q, x], Acta Arith. **22** (1973), 207–220.

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