

On the distribution functions of two oscillating sequences

Christoph Aistleitner* Markus Hofer† Manfred Madritsch‡

Abstract

We investigate the set of all distribution functions of two special sequences on the unit interval, which involve logarithmic and trigonometric terms. We completely characterize the set of all distribution functions $G(x_n)$ for $(x_n)_{n \geq 1} = (\{\cos(\alpha n)^n\})_{n \geq 1}$ and arbitrary α , where $\{x\}$ denotes the fractional part of x . Furthermore we give a sufficient number-theoretic condition on α for which $(x_n)_{n \geq 1} = (\{\log(n) \cos(\alpha n)\})_{n \geq 1}$ is uniformly distributed. Finally we calculate $G(x_n)$ in the case when $\frac{\alpha}{2\pi} \in \mathbb{Q}$.

1 Introduction

In the present paper we consider the set of all distribution functions $G(x_n)$ of sequences $(x_n)_{n \geq 1}, x_n \in [0, 1)$. For an interval $I \subseteq [0, 1)$ we set $A(I, N, x_n)$ to be the number of hits of I among the first N elements of $(x_n)_{n \geq 1}$, i.e.,

$$A(I, N, x_n) = \#\{n \leq N : x_n \in I\} = \sum_{n=1}^N \mathbf{1}_I(x_n).$$

A non-decreasing function $g(x)$ satisfying $g(0) = 0, g(1) = 1$, is called a distribution function of a sequence $(x_n)_{n \geq 1}$ if there exists an increasing sequence $(N_k)_{k \geq 1}$ such that

$$g(x) = \lim_{k \rightarrow \infty} \frac{A([0, x), N_k, x_n)}{N_k}, \quad x \in [0, 1], \quad (1)$$

holds in every point of continuity of g . In the sequel $G(x_n)$ denotes the set of all functions for which (1) holds.

Furthermore a sequence $(x_n)_{n \geq 1}$ is said to have the asymptotic distribution function $g(x)$ if (1) holds for $N_k = k$. Then the set $G(x_n)$ reduces to a singleton. Moreover if $G(x_n) = \{x\}$ the sequence $(x_n)_{n \geq 1}$ is called uniformly distributed (u.d.).

Closely connected to distribution functions of sequences is the concept of the discrepancy of sequences. For a sequence $(x_n)_{n \geq 1}$ the discrepancy of the first N elements of $(x_n)_{n \geq 1}$ is given as

$$D_N((x_n)_{n \geq 1}) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{A([\alpha, \beta), N, x_n)}{N} - (\beta - \alpha) \right|.$$

*Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: aistleitner@math.tugraz.at. Research supported by the Austrian Research Foundation (FWF), Project S9603-N23.

†Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: markus.hofer@tugraz.at. Research supported by the Austrian Research Foundation (FWF), Project S9603-N23 and by the DOC [Doctoral Fellowship Programme of the Austrian Academy of Sciences].

‡Université de Lorraine, Institut Élie Cartan, B.P. 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France. e-mail: manfred.madritsch@univ-lorraine.fr.

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Note that by a theorem of Weyl, see e.g. [11], it follows that $\lim_{N \rightarrow \infty} D_N((x_n)_{n \geq 1}) = 0$ holds if and only if that the sequence $(x_n)_{n \geq 1}$ is u.d. in $[0, 1)$. For general results on uniform distribution of sequences and discrepancy theory see [4], [8] or [11].

The following problem concerning the set of all distribution functions of a sequence is stated in the open problem section on the web site of *Uniform distribution theory*¹:

Find the set $G(x_n)$ for the following sequences:

$$(x_n)_{n \geq 1} = (\{\cos(n)^n\})_{n \geq 1}, \quad (2)$$

$$(x_n)_{n \geq 1} = (\{\log(n) \cos(\alpha n)\})_{n \geq 1}, \quad (3)$$

$$(x_n)_{n \geq 1} = (\{\cos(n + \log(n))\})_{n \geq 1}, \quad (4)$$

where $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$.

For the sequence in (4) this problem has already been solved by S. Steinerberger. One can find a short version of the proof in the open problem section of *Uniform distribution theory*¹. The fact that the sequence in (4) is not u.d. has previously been proved by Kuipers, see [7].

The outline of the rest of the paper is as follows: in the second section we characterize the set of all distribution functions of a general version of the sequence given in (2). We use the Koksma inequality, see e.g. [11], in the third section to find a sufficient condition on the parameter α for which $(x_n)_{n \geq 1}$ given in (3) is u.d. Furthermore we give a complete solution of the problem in the case $\frac{\alpha}{2\pi} \in \mathbb{Q}$.

2 The set of all distribution functions of $(\{\cos(\alpha n)^n\})_{n \geq 1}$

Former results on the sequence (2) are due to Bukor [3] and in more generality Luca [9]. Bukor showed that $(\cos(n)^n)_{n \geq 1}$ is everywhere dense in the interval $[-1, 1]$. The following theorem characterizes the set of distribution functions $G(x_n)$ for

$$(x_n)_{n \geq 1} = (\{\cos(\alpha n)^n\})_{n \geq 1},$$

which is a generalized version of (2).

Theorem 2.1 For $\frac{\alpha}{2\pi} \notin \mathbb{Q}$ we set $a = 3/4$, in the case $\frac{\alpha}{2\pi} = \frac{p}{q} \in \mathbb{Q}$ for p, q co-prime let

$$a = \lim_{N \rightarrow \infty} \frac{\#\{n \leq N : (\cos(\alpha n))^n \geq 0\}}{N} = \begin{cases} \frac{q+1}{2q} + \frac{q-1}{4q}, & \text{if } 4 \mid (q-1), \\ \frac{q-1}{2q} + \frac{q+1}{4q}, & \text{if } 4 \nmid (q-1) \end{cases} \quad (5)$$

for q odd and let

$$\begin{aligned} a &= \lim_{N \rightarrow \infty} \frac{\#\{n \leq N : (\cos(\alpha n))^n \geq 0\}}{N} \\ &= \frac{\#\{1 \leq n \leq q : (\cos(\alpha n))^n \geq 0\}}{q} = \begin{cases} \frac{1}{2} + \frac{q-2}{4q}, & \text{if } 4 \nmid q \text{ and } 8 \mid (q-2), \\ \frac{1}{2} + \frac{q+2}{4q}, & \text{if } 4 \nmid q \text{ and } 8 \nmid (q-2), \\ \frac{q+2}{2q} + \frac{1}{4}, & \text{if } 4 \mid q \text{ and } 8 \nmid q, \\ \frac{q+2}{2q} + \frac{q-4}{4q}, & \text{if } 8 \mid q \end{cases} \end{aligned} \quad (6)$$

¹Problem 1.10 in the open problem collection as of 11. December 2011 (<http://www.boku.ac.at/MATH/udt/unsolvedproblems.pdf>)

for q even.

Then the set of all distribution functions of $(x_n)_{n \geq 1}$ is given by $G(x_n) = \{g_a(x)\}$, with

$$g_a(x) = \begin{cases} 0, & \text{if } x = 0, \\ a, & \text{if } 0 < x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

Proof:

First we consider the case $\frac{\alpha}{2\pi} \notin \mathbb{Q}$. Let $\epsilon, \delta > 0$ be small and fixed. Then for sufficiently large $N_0 \in \mathbb{N}$ we have

$$\lambda(\{x \in [0, 1]: |\cos(2\pi x)^{N_0}| > \delta\}) < \epsilon,$$

where λ denotes the Lebesgue measure. By the fact that $(\{\frac{\alpha}{2\pi}n\})_{n \geq 1}$ is u.d. in $[0, 1]$ if $\frac{\alpha}{2\pi} \notin \mathbb{Q}$ (see e.g. [11]) it follows that

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : |\cos(\alpha n)^n| > \delta\}}{N} < \epsilon.$$

Furthermore since

$$\lambda(\{x \in [0, 1]: \cos(2\pi x) < 0\}) = \frac{1}{2}$$

and by the fact that also $(\{2\frac{\alpha}{2\pi}n\})_{n \geq 1}$ is u.d. in $[0, 1]$ it follows that

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : \cos(\alpha n)^n < 0\}}{N} = \frac{1}{4}.$$

Noting that $|\cos(\alpha n)^n| \leq \delta$ and $\cos(\alpha n)^n < 0$ imply $\{\cos(\alpha n)^n\} \geq 1 - \delta$. Thus we have

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : \cos(\alpha n)^n > \delta\}}{N} \geq \frac{3}{4} - \epsilon$$

and

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : \cos(\alpha n)^n < 1 - \delta\}}{N} \leq \frac{3}{4} + \epsilon$$

Since ϵ and δ are arbitrary, this proves the first part of the theorem.

In the case $\frac{\alpha}{2\pi} = \frac{p}{q} \in \mathbb{Q}$ we easily see that $(\{\cos(\alpha n)\})_{n \geq 1}$ takes only finitely many different values of the form $\cos\left(2\pi\frac{j}{q}\right)$, $j = 1, \dots, q$, which appear periodically with period q .

Consider the case that q is odd. We want to calculate k given as

$$k = \#\left\{j \in \{1, \dots, q\} : \cos\left(2\pi\frac{j}{q}\right) \geq 0\right\}.$$

By $\cos\left(2\pi\frac{q}{q}\right) = 1$ and

$$\cos\left(2\pi\frac{j}{q}\right) = \cos\left(2\pi\frac{q-j}{q}\right), \quad \text{for } j = 1, \dots, q,$$

it follows that $k = 2l - 1$, where l is the maximal value in $\{1, \dots, q\}$ for which

$$2\pi\frac{l}{q} < \frac{\pi}{2} \Leftrightarrow l < \frac{q}{4}.$$

Thus if $4 \mid (q - 1)$ we have $l = \frac{q-1}{4}$ and it follows $\cos\left(2\pi\frac{j}{q}\right) \geq 0$ for $\frac{q+1}{2}$ values of $j = 1, \dots, q$.

Similarly if $4 \nmid (q - 1)$ we have $l = \frac{q-3}{4}$ and $\cos\left(2\pi\frac{j}{q}\right) \geq 0$ for $\frac{q-1}{2}$ values of $j = 1, \dots, q$.

Since q is odd it follows for every $j = 1, \dots, q$ that the term $\cos\left(2\pi\frac{j}{q}\right)$ appears in $(\cos(\alpha n)^n)_{n \geq 1}$ alternating with odd and even exponent, thus for i such that $\cos(\alpha i) < 0$ it follows that

$$\lim_{N \rightarrow \infty} \frac{\#\{n \leq N : \cos(\alpha i)^{q^{n+i}} > 0\}}{N} = \frac{1}{2}.$$

This, together with the above considerations, proves (5).

Assume now that q is even and $4 \nmid q$. Then $\cos(\alpha i) \neq 0$ for $i = 1, \dots, q$ and by similar considerations as above we have $\cos(\alpha i) > 0$ for $\frac{q}{2}$ values of i and $\frac{q}{2}$ is an odd number. Furthermore for every $j = 1, \dots, q$, the exponent of the term $\cos\left(2\pi\frac{j}{q}\right)$ in $(\cos(\alpha n)^n)_{n \geq 1}$ is either always odd or always even. Moreover it is easy to see that the exponents of $\cos\left(2\pi\frac{j}{q}\right)$ and $\cos\left(2\pi\frac{j+1}{q}\right)$ in $(\cos(\alpha n)^n)_{n \geq 1}$ cannot be both even or both odd. Thus the number of j 's such that $\cos\left(2\pi\frac{j}{q}\right) < 0$ and which appear with even exponent is $\frac{q-2}{4}$ when $8 \mid (q-2)$ and $\frac{q+2}{4}$ when $8 \nmid (q-2)$.

In the case $4 \mid q$, it follows that $\cos(\alpha i) = 0$ for two values of i . Thus $\cos(\alpha i) \geq 0$ for $\frac{q+2}{2}$ values of $i = 1, \dots, q$. Formula (6) follows now by the similar arguments as above. This completes the proof. \square

3 The set of all distribution functions of $(\{\log(n) \cos(\alpha n)\})_{n \geq 1}$

Properties of the sequence (3) have been studied by Berend, Boshernitzan and Kolesnik [1]. They proved the denseness of (3) in $[0, 1)$ for arbitrary α . Furthermore they also concluded in [2] that sequences of the form

$$(\{n^a \log^b(n) \cos(2\pi n \alpha)\})_{n \geq 1} \tag{7}$$

are dense in $[0, 1)$, provided α is irrational and either $a > 0$ or $a = 0, b > 0$. Moreover they showed that the sequence in (7) is u.d. for α irrational and either $a > 0$ or $a = 0, b > 1$. Additionally they remarked without proof that the sequence (3) is not u.d. for uncountably many α , see [1].

The next theorem gives a condition on the parameter α which implies the u.d. property for the sequence in (3).

Theorem 3.1 *Let α be such that the discrepancy of the sequence $(z_n)_{n \geq 1} = (\{\frac{\alpha}{2\pi}n\})_{n \geq 1}$ is of asymptotic order $o\left(\frac{1}{\log(N)}\right)$. Then the sequence*

$$(x_n)_{n \geq 1} = (\{\log(n) \cos(\alpha n)\})_{n \geq 1}$$

is u.d.

Remark 3.1 *It is well-known that there is a close connection between the coefficients of the continued fraction expansion of α and the asymptotic order of the discrepancy of $(\{\alpha n\})_{n \geq 1}$. By a classical result of Khintchine [6] from the metric theory of continued fractions, the discrepancy of $(\{\alpha n\})_{n \geq 1}$ satisfies*

$$D_N(\{\alpha\}, \dots, \{\alpha N\}) = \mathcal{O}(N^{-1}(\log N)(\log \log N)^{1+\epsilon})$$

as $N \rightarrow \infty$ for almost all α . Thus the conclusion of Theorem 3.1 holds for almost all α .

Furthermore, if α is badly approximable, and in particular if α is an quadratic irrational such as $\alpha = \sqrt{2}$, then

$$D_N(\{\alpha\}, \dots, \{\alpha N\}) = \mathcal{O}(N^{-1} \log N)$$

as $N \rightarrow \infty$, and thus the conclusion of Theorem 3.1 also holds for $\frac{\alpha}{2\pi} = \sqrt{2}$. For details see [8, p. 125, Th. 3.4].

For the proof of Theorem 3.1 we will use the following Lemma 3.1, which is a special case of the van der Corput lemma. It can be found e.g. in [8, Chapter 1, Section 1, Lemma 2.1].

Lemma 3.1 *Suppose that $\phi(x)$ is real-valued, that $|\phi'(x)| \geq \gamma$ for some positive γ , and that ϕ' is monotonic for all $x \in (\alpha, \beta)$. Then*

$$\left| \int_{\alpha}^{\beta} \cos(2\pi\phi(x)) dx \right| \leq \gamma^{-1} \quad \text{and} \quad \left| \int_{\alpha}^{\beta} \sin(2\pi\phi(x)) dx \right| \leq \gamma^{-1}.$$

Proof of Theorem 3.1:

To prove uniform distribution of $(x_n)_{n \geq 1}$, by the Weyl criterion (see [8]) it is sufficient to show that

$$\lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \cos(2\pi h x_n) \right| = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \sin(2\pi h x_n) \right| = 0 \quad (8)$$

holds for all $h > 0$. We will only prove (8) for cosine-functions, the case of sine-functions being entirely similar. Assume that $h > 0$ is fixed. Choose $\epsilon > 0$ and $\delta > 0$ “small”, and set

$$K = \frac{-\log(\epsilon)}{\log(1 + \delta)}.$$

For simplicity of writing, we assume that K is an integer. For $N \geq 1$, we define

$$m_i = \lceil \epsilon N (1 + \delta)^i \rceil, \quad 0 \leq i \leq K,$$

and

$$y_n = \log(\epsilon N (1 + \delta)^i) \cos(\alpha n) \quad \text{for} \quad m_{i-1} < n \leq m_i, \quad 1 \leq i \leq K.$$

Note that $m_0 = \lceil \epsilon N \rceil$ and $m_K = \epsilon N (1 + \delta)^K = N$. We have

$$\left| \frac{1}{N} \sum_{n=1}^N \cos(2\pi h x_n) \right| \quad (9)$$

$$= \left| \frac{1}{N} \left(\sum_{n=1}^{m_0} \cos(2\pi h x_n) \right) + \frac{1}{N} \left(\sum_{n=m_0+1}^N (\cos(2\pi h x_n) - \cos(2\pi h y_n) + \cos(2\pi h y_n)) \right) \right|$$

$$\leq \left| \frac{1}{N} \sum_{n=1}^{m_0} \cos(2\pi h x_n) \right| + \left| \frac{1}{N} \sum_{n=m_0+1}^N (\cos(2\pi h x_n) - \cos(2\pi h y_n)) \right| \quad (10)$$

$$+ \left| \frac{1}{N} \sum_{n=m_0+1}^N \cos(2\pi h y_n) \right|. \quad (11)$$

The first term in line (10) is trivially bounded by $m_0/N = \lceil \epsilon N \rceil / N$. Now we turn to the second term in (10). For any $i \in \{1, \dots, K\}$ and $m_{i-1} < n \leq m_i$ we have

$$\begin{aligned} |\cos(2\pi h x_n) - \cos(2\pi h y_n)| &= |\cos(2\pi h u_n) - \cos(2\pi h y_n)| \\ &\leq 2\pi h |u_n - y_n| \\ &= 2\pi h |\log(n) \cos(\alpha n) - \log(\epsilon N (1 + \delta)^i) \cos(\alpha n)| \end{aligned}$$

$$\begin{aligned}
&\leq 2\pi h \left(\lceil \log(\epsilon N(1+\delta)^i) \rceil - \log(n) \right) |\cos(\alpha n)| \\
&\leq 2\pi h \left(\lceil \log(\epsilon N(1+\delta)^i) \rceil - \lceil \log(\epsilon N(1+\delta)^{i-1}) \rceil - 1 \right) \\
&\leq 2\pi h \log(1+\delta).
\end{aligned}$$

where $u_n = \log(n) \cos(\alpha n)$. Thus we get

$$\left| \frac{1}{N} \left(\sum_{n=\epsilon N+1}^N (\cos(2\pi h x_n) - \cos(2\pi h y_n)) \right) \right| \leq (2\pi h)^2 \log(1+\delta). \quad (12)$$

Finally we estimate the term in line (11). For $1 \leq i \leq K$ we set

$$f_i(x) = \cos \left(2\pi h \left(\log(\epsilon N(1+\delta)^i) \cos(2\pi x) \right) \right).$$

The total variation $\text{Var}(f_i)$ of this function on the interval $[0, 1]$ is at most

$$8h \lceil \log(\epsilon N(1+\delta)^i) \rceil. \quad (13)$$

For any number $R > 1$, in the interval $[R^{-1/2}, 1/4 - R^{-1/2}]$ the derivative of the function $R \cos(2\pi x)$ is monotonous, and of absolute value at least $R^{1/2}/2$. Thus, using Lemma 3.1 for the function $f_i(x) = \cos(2\pi \Phi(x))$, where $\Phi(x) = R \cos(2\pi x)$ and the previous remark for $R = h(\log(\epsilon N(1+\delta)^i))$, we have

$$\begin{aligned}
\left| \int_0^{1/4} f_i(x) dx \right| &\leq \underbrace{\int_0^{R^{-1/2}} |f_i(x)| dx + \int_{1/4-R^{-1/2}}^{1/2} |f_i(x)| dx}_{\leq 2R^{-1/2}} + \underbrace{\left| \int_{R^{-1/2}}^{1/4-R^{-1/2}} f_i(x) dx \right|}_{\leq 2R^{-1/2}} \\
&\leq 4h^{-1/2} (\log(\epsilon N(1+\delta)^i))^{-1/2}.
\end{aligned}$$

A similar estimate holds for the absolute value of $\int_{i/4}^{(i+1)/4} f_i(x) dx$, $i = 1, 2, 3$, and in total we obtain

$$\left| \int_0^1 f_i(x) dx \right| \leq 16h^{-1/2} (\log(\epsilon N(1+\delta)^i))^{-1/2} \leq 16(\log(\epsilon N))^{-1/2}, \quad 1 \leq i \leq K. \quad (14)$$

For the term in line (11) we have

$$\left| \frac{1}{N} \left(\sum_{n=\lceil \epsilon N \rceil+1}^N \cos(2\pi h y_n) \right) \right| \leq \sum_{i=1}^K \frac{m_i - m_{i-1}}{N} \left| \frac{1}{m_i - m_{i-1}} \left(\sum_{n=m_{i-1}+1}^{m_i} \cos(2\pi h y_n) \right) \right|. \quad (15)$$

Note that we can write

$$\cos(\alpha n) = \cos \left(2\pi \left(\left\lfloor \frac{\alpha n}{2\pi} \right\rfloor + \left\{ \frac{\alpha n}{2\pi} \right\} \right) \right) = \cos \left(2\pi \left\{ \frac{\alpha n}{2\pi} \right\} \right) = \cos(2\pi z_n).$$

Thus for any i , $1 \leq i \leq K$, using Koksma's inequality for the function $f_i(x)$, together with (13) and (14), we get

$$\begin{aligned}
&\left| \frac{1}{m_i - m_{i-1}} \left(\sum_{n=m_{i-1}+1}^{m_i} \cos(2\pi h y_n) \right) \right| \\
&= \left| \frac{1}{m_i - m_{i-1}} \left(\sum_{n=m_{i-1}+1}^{m_i} \cos(2\pi h \log(\epsilon N(1+\delta)^i) \cos(2\pi z_n)) \right) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \left| \int_0^1 f_i(x) dx \right| + \text{Var}(f_i) D_{m_i - m_{i-1}}(z_{m_{i-1}+1}, \dots, z_{m_i}) \\
&\leq 16(\log(\epsilon N))^{-1/2} + 8h \lceil \log(\epsilon N(1 + \delta)^i) \rceil D_{m_i - m_{i-1}}(z_{m_{i-1}+1}, \dots, z_{m_i}).
\end{aligned} \tag{16}$$

By the triangle inequality for discrepancies and the assumption on the discrepancy of $(z_n)_{n \geq 1}$,

$$\begin{aligned}
&D_{m_i - m_{i-1}}(z_{m_{i-1}+1}, \dots, z_{m_i}) \\
&\leq \frac{m_{i-1}}{m_i - m_{i-1}} D_{m_{i-1}}(z_1, \dots, z_{m_{i-1}}) + \frac{m_i}{m_i - m_{i-1}} D_{m_i}(z_1, \dots, z_{m_i}) \\
&\leq \frac{\epsilon}{\log N}
\end{aligned} \tag{17}$$

for all $i \in \{1, \dots, K\}$, provided N is sufficiently large. Using (16) and (17), we thus see that (15) is at most

$$16(\log(\epsilon N))^{-1/2} + 8h\epsilon \frac{\lceil \log N \rceil}{\log N} \tag{18}$$

for sufficiently large N .

Combining all our estimates for (9), we finally obtain

$$\left| \frac{1}{N} \sum_{n=1}^N \cos(2\pi h x_n) \right| \leq \frac{\lceil \epsilon N \rceil}{N} + (2\pi h)^2 \log(1 + \delta) + 16(\log(\epsilon N))^{-1/2} + 8h\epsilon \frac{\lceil \log N \rceil}{\log N}$$

for sufficiently large N . Since ϵ and δ can be chosen arbitrarily small, this proves the theorem. \square

The following lemma by Pólya and Szegő [10] characterizes the set of distribution functions $G(\{c \log(n)\})$ and is the main tool for the proof of Theorem 3.2.

Lemma 3.2 (Pólya and Szegő) *The sequence $(x_n)_{n \geq 1} = (\{c \log n\})_{n \geq 1}$, $c > 0$, has distribution functions of the form*

$$g_{\beta, c}(x) = \frac{e^{\frac{\min(x, \beta)}{c}} - 1}{e^{\frac{\beta}{c}} - 1} + \frac{1}{e^{\frac{\beta}{c}}} \frac{e^{\frac{x}{c}} - 1}{e^{\frac{1}{c}} - 1}, \tag{19}$$

where $\lim_{k \rightarrow \infty} \{c \log N_k\} = \beta$ implies $F_{N_k}(x) \rightarrow g_{\beta}(x)$ and $F_N(x) = \frac{\#\{n \leq N; x_n \in [0, x]\}}{N}$. Moreover $G(\{c \log(n)\})$ is the set of all distribution functions of the form (19).

Remark 3.2 *Note that it follows as a corollary of Lemma 3.2 that the sequence $(x_n)_{n \geq 1} = (\{c \log(n)\})_{n \geq 1}$, $c < 0$, has distribution functions of the form $1 - g_{\beta, |c|}(1 - x)$, where $\lim_{k \rightarrow \infty} \{|c| \log N_k\} = \beta$ and $g_{\beta, |c|}(x)$ is given in (19). Thus we define the function $f_{\beta, c}(x)$ as*

$$f_{\beta, c}(x) = \begin{cases} g_{\beta, c}(x), & \text{if } c > 0, \\ 1 - g_{\beta, |c|}(1 - x) & \text{if } c < 0, \\ \mathbf{1}_{\{(0, 1]\}}(x) & \text{if } c = 0, \end{cases} \tag{20}$$

where $\lim_{k \rightarrow \infty} \{|c| \log N_k\} = \beta$.

Theorem 3.2 *Let $\frac{p}{q} := \frac{\alpha}{2\pi} \in \mathbb{Q}$ where p, q are co-prime and*

$$(x_n)_{n \geq 1} = (\{\log(n) \cos(\alpha n)\})_{n \geq 1}.$$

Then for a fixed subsequence $(N_k)_{k \geq 1}$ of \mathbb{N} with

$$\lim_{k \rightarrow \infty} \{\cos(\alpha i) \log N_k\} = \beta_i, \quad \text{for } i = 1, \dots, q, \quad (21)$$

the asymptotic distribution of $(x_n)_{n \geq 1}$ along the subsequence $(N_k)_{k \geq 1}$ is given by

$$f(x) = \frac{1}{q} \sum_{i=1}^q h_{q, \beta_i, c_i}(x), \quad (22)$$

where $h_{q, \beta_i, c_i}(x)$ is given in (20),

$$h_{q, \beta_i, c_i}(x) = \begin{cases} f_{\beta_i, c_i}(x + 1 - \nu_i) - f_{\beta_i, c_i}(1 - \nu_i), & \text{if } 0 \leq x \leq \nu_i \text{ and } c_i > 0, \\ f_{\beta_i, c_i}(x - \nu_i) + 1 - f_{\beta_i, c_i}(1 - \nu_i), & \text{if } \nu_i \leq x \leq 1 \text{ and } c_i > 0, \\ f_{\beta_i, c_i}(x + \nu_i) - f_{\beta_i, c_i}(\nu_i), & \text{if } 0 \leq x \leq 1 - \nu_i \text{ and } c_i < 0, \\ f_{\beta_i, c_i}(x - (1 - \nu_i)) + 1 - f_{\beta_i, c_i}(\nu_i), & \text{if } 1 - \nu_i \leq x \leq 1 \text{ and } c_i < 0, \\ \mathbf{1}_{\{(0,1]\}}(x) & \text{if } c_i = 0, \end{cases} \quad (23)$$

where $\nu_i = \{|c_i| \log(q)\}$, $c_i = \cos(\alpha i)$, $\lim_{n \rightarrow \infty} \{\cos(\alpha i) \log N_k\} = \beta_i$ and $f_{\beta, c}(x)$ is given in (20). Moreover, the set $G(x_n)$ is the set of all distribution functions of the form (22) for those $(\beta_1, \dots, \beta_q)$ for which a subsequence $(N_k)_{k \geq 1}$ satisfying (21) exists.

Remark 3.3 For arbitrary q , it is a difficult problem to determine all possible vectors $(\beta_1, \dots, \beta_q)$ for which there exists a subsequence $(N_k)_{k \geq 1}$ for which (21) holds, because there can exist non-trivial linear relations between the values $\cos(\alpha i)$, $i = 1, \dots, q$ (depending on number-theoretic properties of q). We will not further investigate this issue, the interested reader is referred to a Galois theoretic approach to this problem by Girstmair [5].

Proof:

As mentioned in the proof of Theorem 2.1 the function $\cos(\alpha n)$, $n \in \mathbb{N}$ takes only finitely many different values which appear in a period of length q . Let $(N_k)_{k \geq 1}$ be a sequence of \mathbb{N} for which (21) holds for some numbers β_1, \dots, β_q . We are interested in the asymptotic behavior of $(x_n)_{n \geq 1}$ and by

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{A([a, b], N_k, x_n)}{N_k} &= \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^q A([a, b], \lfloor N_k/q \rfloor, \cos(\alpha i) \log(qn + i))}{N_k} \\ &= \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^q A([a, b], \lfloor N_k/q \rfloor, \cos(\alpha i) \log(qn))}{N_k}, \end{aligned}$$

we get that the limit distribution of $(x_n)_{n \geq 1}$ is a mixture of limit distributions of q sequences of the form

$$(z_n^i)_{n \geq 1} = (\{\log(qn) \cos(\alpha i)\})_{n \geq 1}, \quad \text{for } i \in \{1, \dots, q\}.$$

Since $\log(qn) \cos(\alpha i) = \log(q) \cos(\alpha i) + \log(n) \cos(\alpha i)$ we get that the distribution function of $(z_n^i)_{n \geq 1}$ is given as the distribution function of $\log(n) \cos(\alpha i)$ shifted by a constant and thus it is easy to see that the right hand-side of (23) is the distribution function of $(z_n^i)_{n \geq 1}$. This proves (22).

In order to prove that all functions in $G(x_n)$ can be characterized by (22) we use the Bolzano-Weierstrass theorem. Assume that $(\{\cos(\alpha i) \log N_k\})_{k \geq 1}$ does not converge for at least one i , then it follows that there are at least two convergent subsequences of $(\{\cos(\alpha i) \log N_k\})_{k \geq 1}$ which have different limits. If the limit distributions along all these subsequences are equal then the limit distribution of $(\{\cos(\alpha i) \log N_k\})_{k \geq 1}$ along the whole sequence $(N_k)_{k \geq 1}$ is (if it exists) can be written as $f_{\beta, c}(h(x))$ as the limit along a sequence for which (21) holds. If at least two limit distributions along subsequences are not equal the limit distribution along $(\{\cos(\alpha i) \log N_k\})_{k \geq 1}$ does not exist. \square

Remark 3.4 We illustrate the set $G(x_n)$ in the simple case when $\alpha = \pi$, which means that $p = 1$ and $q = 2$ in the notation of Theorem 3.2. Note that $\cos(\alpha) = -1$ and $\cos(2\alpha) = 1$, thus as a sequence $(N_k)_{k \geq 1}$ which satisfies (21) one can choose for example $N_k = \lfloor \exp(k + \beta_1) \rfloor$. In this case we have

$$\lim_{k \rightarrow \infty} \{\cos(\alpha) \log(N_k)\} = \beta_1$$

and

$$\lim_{k \rightarrow \infty} \{\cos(2\alpha) \log(N_k)\} = \beta_2 = 1 - \beta_1.$$

We can calculate the corresponding distribution function by using the previous theorem. Figure 1 illustrates the range of distribution functions which one can achieve by varying β .

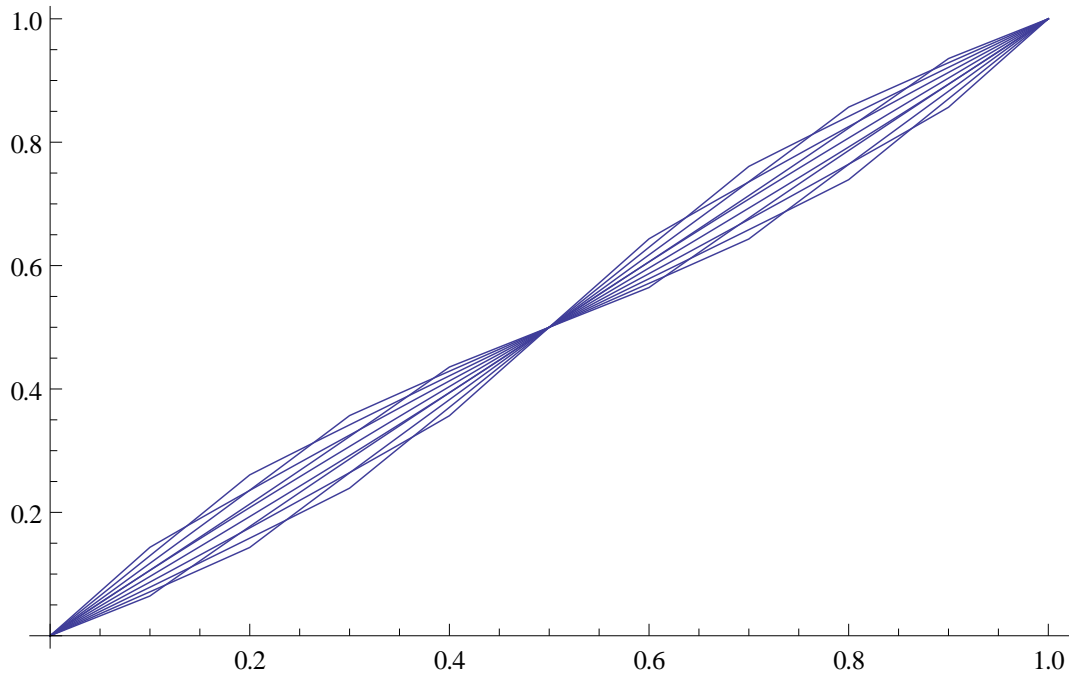


Figure 1: Distribution functions of $(x_n)_{n \geq 1}$ for $\alpha = \pi$ and $\beta = 0, \frac{1}{10}, \dots, \frac{9}{10}$.

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