ON SIMULTANEOUS DIGITAL EXPANSIONS OF POLYNOMIAL VALUES

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ABSTRACT. Let s_q denote the q-ary sum-of-digits function and let $P_1(X)$, $P_2(X) \in \mathbb{Z}[X]$ with $P_1(\mathbb{N}), P_2(\mathbb{N}) \subset \mathbb{N}$ be polynomials of degree $h, l \geq 1$, $h \neq l$, respectively. In this note we show that $(s_q(P_1(n))/s_q(P_2(n)))_{n \geq 1}$ is dense in $\mathbb{R}^+$. This extends work by Stolarsky (1978) and Hare, Laishram and Stoll (2011).

1. INTRODUCTION

Let $q \geq 2$. Then we can express $n \in \mathbb{N}$ uniquely in base q as

$$n = \sum_{j \geq 0} n_j q^j, \qquad n_j \in \{0, 1, \dots, q-1\}. \tag{1}$$

Denote by $s_q(n) = \sum_{j \geq 0} n_j$ the sum of digits of n in base q. The sum of digits of polynomial values has been at the center of interest in many works. We mention the (still open) conjecture of Gelfond [5] from 1967/68 about the distribution of s_q of polynomial values in arithmetic progressions (see also [4, 7, 10]) and the fundamental work of Bassily and Kátai [1] on central limit theorems satisfied by s_q supported on polynomial values resp. polynomial values with prime arguments.

In 1978, Stolarsky [9] examined the pointwise relationship between $s_q(n^h)$ and $s_q(n)$, where $h \geq 2$ is a fixed integer. In particular, he used a result of Bose and Chowla [2] to prove that

$$\limsup_{n \to \infty} \frac{s_2(n^h)}{s_2(n)} = \infty. \tag{2}$$

Hare, Laishram and Stoll [6] generalized (2) to an arbitrary polynomial $P(X) \in \mathbb{Z}[X]$ of degree $h \geq 2$ in place of X^h, and to base q in place of the binary base. Moreover, they showed that on the other side of the spectrum,

$$\liminf_{n \to \infty} \frac{s_q(P(n))}{s_q(n)} = 0, \tag{3}$$

confirming a conjecture of Stolarsky.

From another point of view, not much is known about the pointwise relationship between the sum-of-digits of the values of two distinct fixed

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integer-valued polynomials $P_1(X)$, $P_2(X)$. Building up on work by Drmota [3], Steiner [8] showed that the distribution of the two-dimensional vector $(s_q(P_1(n)), s_q(P_2(n)))_{n\geq 1}$ obeys a central limit law (in fact, his results apply for general vectors and general q-additive functions). However, there are no local results available, such as an asymptotic formula or even a non-trivial lower estimate for

$$\#\{n < x : \quad s_q(P_1(n)) = k_1, \quad s_q(P_2(n)) = k_2\}, \tag{4}$$

where k_1 and k_2 are fixed positive integers.

The purpose of the present work is to extend both (2) and (3), and to make a first step towards understanding (4).

Our main result is as follows:

Theorem 1.1. *Let $P_1(X), P_2(X) \in \mathbb{Z}[X]$ be polynomials of distinct degrees $h, l \geq 1$ with $P_1(\mathbb{N}), P_2(\mathbb{N}) \subset \mathbb{N}$. Then*

$$\left(\frac{s_q(P_1(n))}{s_q(P_2(n))}\right)_{n\geq 1}$$

is dense in $\mathbb{R}^+$.

Remark 1. The proof extends to strictly q-additive functions in place of the sum-of-digits function s_q (we need, however, the condition that the weight attached to the digit $q-1$ is positive, *cf.* (19)). Recall that a strictly q-additive function f is a real-valued function f defined on the non-negative integers which satisfies $f(0) = 0$ and $f(n) = \sum_{j\geq 0} f(n_j)$, where the n_j are the digits in the q-adic expansion (*cf.* (1)).

We first state some notation that is used throughout the paper. For integers a, b with $b < a$ we will write $[b, a]$ for the set of integers $\{b, b+1, \ldots, a\}$. For sets A and B, we write $mA = \{a_1 + \cdots + a_m : \ a_i \in A, \ 1 \leq i \leq m\}$ and $A + B = \{a + b : \ a \in A, \ b \in B\}$. For the sake of simplicity, we allow all constants to depend on q without further mentioning. Since we fix q already in the beginning there is not much harm to do so.

2. Proof of the main result

The proof of Theorem 1.1 will proceed in several steps. We first address the case $P_1(X) = X^h$, $P_2(X) = X^l$ which can be dealt with in a well arranged manner. The key idea in the proof is that $s_q(q^u) = 1$ whereas $s_q(q^u - 1) = (q-1)u$, so that the first value is independent of u (and negligable, as $u \to \infty$) and the second one increases as u increases. In order to exploit this, we construct in Section 2.1 a polynomial $p(X)$ and determine the number of negative coefficients in $p(X)^t$ for $t \geq 2$. In Section 2.2 we then show that, given a real number $r \in (0, 1)$, we can choose the parameters of the polynomial in such a way that the ratios of the numbers of negative coefficients of $p(X)^h$ and $p(X)^l$ approximate r arbitrarily well. In Section 2.3 we link this ratio to the ratio of the sum-of-digits function under question and show that we obtain the same limit. The final two sections concern the generalization to arbitrary $r \in \mathbb{R}^+$ and to arbitrary polynomials $P_1(X), P_2(X)$, respectively.

2.1. **Construction of the polynomial** $p(X)$**.** In this section we construct the polynomial $p(X)$ which we will use later to approximate a given positive real ratio r. Let $a, b, c, d, e \in \mathbb{N}$ with $a > b > c \geq d > e > 0$ and set

$$A_1 = [0, e], \quad A_2 = [d, c], \quad A_3 = [b, a].$$

Let $k \in \mathbb{N}$ and define[1] the polynomial $p(X) \in \mathbb{Z}[X]$ by

$$p(X) = q^k \sum_{i \in A_1} X^i - \sum_{i \in A_2} X^i + q^k \sum_{i \in A_3} X^i. \tag{5}$$

The reason for putting these weights to the powers X^i is our simple wish to control the number of negative coefficients in the expansion of $p(X)^t$. In fact, we will choose k in such a way that in the expansion of the power $p(X)^t$ the terms that only use the coefficients q^k dominate over those terms that involve -1 terms in the product. Later on, we will evaluate $p(X)^t$ at $X = q^u$ for some large u, so we also need to have good control on the sum of digits of this value. To achieve this goal, we suppose that A_1 and A_3 have the same size and that the set A_2 lies symmetric around $a/2$. More precisely, we suppose that

$$e = a - b \quad \text{and} \quad a = c + d. \tag{6}$$

Denote by $t \geq 2$ a fixed integer. We now look at the sign structure of the coefficients in the expansion of $p(X)^t$. For $0 \leq i \leq t$ set

$$Q_i = iA_1 + (t-i)A_3 = [(t-i)b,\ ie + (t-i)a] = [(t-i)b,\ ta - ib]. \tag{7}$$

By (6) and (7) the sets Q_i, $1 \leq i \leq t$, are pairwise disjoint provided that

$$\frac{a}{b} < \frac{t+1}{t}. \tag{8}$$

We note that the function on the right hand side is decreasing in t. We claim that there exists an integer $k_0(t, a) > 0$ such that for $k \geq k_0(t, a)$ all the coefficients of $p(X)^t$ of the powers

$$X^m \quad \text{with} \quad m \in \bigcup_{i=0}^{t} Q_i$$

are positive. To see this, we note that the positive coefficients of $p(X)$ that contribute to X^m have total weight at least q^{tk}, whereas the total contribution to X^m of terms that involve at least one negative coefficient of $p(X)$ is $O_{t,a}(q^{(t-1)k})$ (where the implied constant depends on t and a). In other terms, for each t and a there is $k_0(t, a) > 0$ such that for all $k \geq k_0(t, a)$ the coefficients of $p(X)^t$ belonging to powers of the sets Q_i are positive.

[1] It is worth mentioning that the degree four polynomial used in [6] is exactly of this form with $(a, b, c, d, e) = (4, 3, 2, 2, 1)$.

For $0 \le i \le t-1$, we call

$$G_i = [1 + \max Q_{i+1},\ -1 + \min Q_i] = [ta - (i+1)b + 1,\ (t-i)b - 1]$$

the *gap* between Q_{i+1} and Q_i. Each G_i contains

$$(t-i)b - 1 - (ta - (i+1)b + 1) + 1 = (t+1)b - ta - 1 \tag{9}$$

integers. This quantity is independent of i. Our aim is to determine sufficient conditions under which all the gaps in the expansion of $p(X)^t$ are "filled" with powers having negative coefficients. Otherwise said, we want that

$$X^m \quad \text{with} \quad m \in \bigcup_{i=0}^{t-1} G_i$$

all have negative coefficients. Since the Q_i's are disjoint, for each X^m with $m \in \bigcup_{i=0}^{t-1} G_i$ there must be a contributing term that involves at least one coefficient attached to some power with exponent in A_2. We use a similar argument as above: The total contribution from coefficients that involve ≥ 2 terms from A_2 is $O_{t,a}(q^{(t-2)k})$. On the other hand, the total weight of those contributions that involve exactly one coefficient from A_2 is (in modulus) at least $q^{(t-1)k}$. Therefore, there exists an integer $k_1(t,a) > 0$ such that for all $k \ge k_1(t,a)$ the contributions of the terms that use exactly one term from A_2 are dominating.

The negative coefficients originate from terms that use one, three etc. factors with exponents in A_2. According to the previous reasoning, it is sufficient to consider only those negative coefficients that use just one single factor. We define

$$N_i = iA_1 + A_2 + (t-1-i)A_3, \quad 0 \le i \le t-1.$$

Again by (6), this simplifies to

$$N_i = [(a-c) + (t-1-i)b,\ i(a-b) + c + (t-1-i)a]. \tag{10}$$

As indicated above, we will completely fill the gaps between two blocks of positive coefficients by negative ones. To achieve this goal, we suppose that for all i with $0 \le i \le t-1$,

$$\max N_i \ge \min Q_i \quad \text{and} \quad \max Q_{i+1} \ge \min N_i.$$

It is a straightforward calculation that both inequalities reduce to the same inequality, namely,

$$(t-1)a + c \ge tb, \tag{11}$$

which is independent of i.

Denote by $C_t(p)$ the number of negative coefficients in the expansion of $p(X)^t$. We first want to show that for $1 \le l < h$ and each $r \in \mathbb{R}$ with $r \in (0,1)$ there exists (a,b,c,d,e) such that $C_h(p)/C_l(p)$ is "close" to r. In fact, our construction will yield an infinite sequence of quintuples (a,b,c,d,e) such that the ratio is arbitrarily close to r. A simple observation then gives the result for any real number $r \in \mathbb{R}^+$ as well as for $1 \le h < l$.

As for now, let us assume that $2 \leq l < h$. Since we are interested in the ratio $C_h(X)/C_l(X)$ we apply the reasoning from above to $p(X)^h$ and $p(X)^l$. The condition (11) gives

$$(h-1)a + c \geq hb \quad \text{and} \quad (l-1)a + c \geq lb. \tag{12}$$

Combining the inequalities (12) with (8) and (6) we get that all gaps are filled with powers having negative coefficients provided that

$$\frac{a}{b} < \frac{h+1}{h} < \frac{l+1}{l} \tag{13}$$

and

$$b > c \geq \max\left(\frac{a}{2},\ hb-(h-1)a,\ lb-(l-1)a\right). \tag{14}$$

We want to find sufficient conditions under which this interval for c is not empty. First, $a/2 \leq b-1$ is equivalent to $\frac{a}{b} \leq 2-\frac{2}{b}$ and (13) is stronger than this inequality provided that $b \geq 4$. On the other hand, we automatically have $hb-(h-1)a \leq b-1$ and $lb-(l-1)a \leq b-1$ because $h, l \geq 2$ and $a > b$. Summing up, whenever

$$a > b \geq 4 \quad \text{and} \quad a/b < (h+1)/h,$$

we can find c and therefore by our restrictions in (6) also d and e such that all gaps in $p(X)^h$ and $p(X)^l$ are filled with powers having negative coefficients.

The same reasoning can be used to deal with $l = 1$. In this case, however, we are forced to take $c = b-1$. There is no additional condition such as (13) or (14), and the gap is completely filled.

2.2. The number of negative coefficients of $p(X)^h$ and $p(X)^\ell$.

Again let us first assume that $2 \leq l < h$. Now we count the numbers of powers that have negative coefficients in the expansions of $p(X)^h$ and $p(X)^l$, respectively. There are exactly h resp. l gaps in $p(X)^h$ resp. $p(X)^l$. Therefore, by (9) and (13),

$$C_t(p) = t((t+1)b - ta - 1), \qquad t \geq 2,$$

and

$$\frac{C_h(p)}{C_l(p)} = \frac{h}{l} \cdot \frac{(h+1) - h\,\frac{a}{b} - \frac{1}{b}}{(l+1) - l\,\frac{a}{b} - \frac{1}{b}}.$$

We consider the function $f : \mathbb{R} \setminus \{(l+1)/l\} \to \mathbb{R}$,

$$f(x) = \frac{h}{l} \cdot \frac{(h+1) - hx}{(l+1) - lx}.$$

A first observation is that

$$f\left(\frac{h+l+1}{h+l}\right) = 1 \quad \text{and} \quad f\left(\frac{h+1}{h}\right) = 0,$$

and that for all $x \neq \frac{l+1}{l}$,

$$f'(x) = -\frac{h(h-l)}{l(lx-l-1)^2} < 0.$$

Thus, f is non-negative and strictly decreasing on the real interval

$$I = \left[\frac{h+l+1}{h+l}, \frac{h+1}{h}\right]$$

and, in particular, $\eta := \max_{x\in I} |f'(x)| > 0$. Let $r \in (0,1)$ and $\varepsilon > 0$ be given real numbers. Since f is uniformly continuous on I there exists $b_0 = b_0(\varepsilon) \geq 4$ such that for all $b \geq b_0$ and $x \in I$,

$$\left| f(x) - \frac{h}{l} \cdot \frac{(h+1) - hx - \frac{1}{b}}{(l+1) - lx - \frac{1}{b}} \right| < \frac{\varepsilon}{4}. \tag{15}$$

Let $\xi \in I$ be the unique real number such that $f(\xi) = r$. Then there exist integers a, b with $a > b \geq b_0(\varepsilon)$ and $a/b \in I$ such that

$$\left| \xi - \frac{a}{b} \right| < \frac{\varepsilon}{4\eta}. \tag{16}$$

(In fact there is an infinity of such pairs (a,b); note that we do not restrict a and b to coprime integers.) From now on, let a and b be fixed integers that satisfy (16). We then choose an integer c in the interval given by (14) (for instance, $c = b-1$ is an admissible value).

Next, we turn to the ratio $C_h(p)/C_l(p)$. The inequalities (15) and (16) yield

$$\begin{aligned} \left| \frac{C_h(p)}{C_l(p)} - r \right| &\leq \left| \frac{C_h(p)}{C_l(p)} - f\left(\frac{a}{b}\right) \right| + \left| f\left(\frac{a}{b}\right) - f(\xi) \right| \\ &< \frac{\varepsilon}{4} + \eta \cdot \left| \xi - \frac{a}{b} \right| < \frac{\varepsilon}{2}. \end{aligned} \tag{17}$$

In the case of $l = 1$ we get $C_1(p) = 2b - a - 1$ and the same argument applies.

2.3. **A first approximation.** From now on, suppose that $1 \leq l < h$. We turn our attention to the ratio $s_q(n^h)/s_q(n^l)$. We show that for $n = p(q^u)$ and $u \to \infty$, we have

$$s_q(n^h)/s_q(n^l) \to C_h(p)/C_l(p).$$

Let

$$k > \max\left(k_0(h,a), k_0(l,a), k_1(h,a), k_1(l,a)\right)$$

and take it to be a fixed value. For $\varepsilon > 0$ and $r \in (0,1)$ we have constructed in the previous section a concrete polynomial $p(X)$ of the form (5) that satifies (17).

Since by now $p(X)$ is fixed and only depends on r, ε, h and l, we have also that there is $\Delta = \Delta(r, \varepsilon, h, l) > 0$ such that

$$\max\left(\max_{0\leq m\leq ah} \left| [X^m]p(X)^h \right|, \max_{0\leq m\leq al} \left| [X^m]p(X)^l \right| \right) < \Delta. \tag{18}$$

Obviously, there exists $u_0 = u_0(r, \varepsilon, h, l)$ such that for all $u \geq u_0$ we have $q^u > \Delta$. We now use the splitting property of the sum-of-digits function (see [6, Proposition 2.1] for a proof): For all $m_1, u \geq 1$ and $1 \leq m_2 < q^u$ we have

$$
\begin{aligned}
(19)\qquad s_q(m_1 q^u + m_2) &= s_q(m_1) + s_q(m_2),\\
s_q(m_1 q^u - m_2) &= s_q(m_1 - 1) + (q-1)u - s_q(m_2 - 1).
\end{aligned}
$$

By (18), we can successively apply (19) to the terms in the expansion of $p(q^u)^h$ and $p(q^u)^l$, respectively (It is sufficient to observe that by our choice of k each coefficient is larger than or equal to 1 in modulus and smaller than Δ in modulus). This yields

$$
(20)\qquad \frac{s_q(p(q^u)^h)}{s_q(p(q^u)^l)} = \frac{C_h(p)u(q-1) + M_1(r,\varepsilon,h,l)}{C_l(p)u(q-1) + M_2(r,\varepsilon,h,l)},
$$

where $M_1 = M_1(r,\varepsilon,h,l)$ and $M_2 = M_2(r,\varepsilon,h,l)$ are independent of u for $u \geq u_0$. Moreover, for the given $\varepsilon > 0$ there exists $u_1 = u_1(r,\varepsilon,h,l) > 0$ such that for all $u \geq u_1$,

$$
\left| \frac{C_h(p) + \frac{M_1}{u(q-1)}}{C_l(p) + \frac{M_2}{u(q-1)}} - \frac{C_h(p))}{C_l(p)} \right| < \frac{\varepsilon}{2}.
$$

Now, choose $u \geq \max(u_0, u_1)$. Then, again by the triangle inequality, we get

$$
\begin{aligned}
\left| \frac{s_q(p(q^u)^h)}{s_q(p(q^u)^l)} - r \right| &\leq \left| \frac{C_h(p) + \frac{M_1}{u(q-1)}}{C_l(p) + \frac{M_2}{u(q-1)}} - \frac{C_h(p))}{C_l(p)} \right| + \left| \frac{C_h(p)}{C_l(p)} - r \right|\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

This already proves that for $1 \leq l < h$ the sequence $\left(s_q(n^h)/s_q(n^l)\right)_{n\geq 1}$ lies dense in $(0,1)$.

2.4. **The generalization to arbitrary $r \in \mathbb{R}^+$.** We now show how to approximate $r \in \mathbb{R}^+$ that lie outside of $(0,1)$. Recall that we still assume that $1 \leq l < h$. Denote by $\nu = \nu(r,h,l)$ the minimal positive integer such that

$$
(21)\qquad r_0 = r \cdot \left(\frac{h+1}{l+1}\right)^{-\nu} \in (0,1).
$$

We construct $p(X)$ in the same manner as before, where r is replaced by r_0 and ε by $\varepsilon \left(\frac{h+1}{l+1}\right)^{-\nu}$. Now, define

$$
p_0(X) = p(X), \qquad p_{i+1}(X) = p_i(X)(1 + X^{w_i}), \quad 0 \leq i \leq \nu - 1,
$$

where $w_i = w_i(p_i, \nu)$ is a large integer. Then it is easy to see that $C_h(p_{i+1}) = 2(h+1)C_h(p_i)$ and that $C_h(p_\nu) = 2^\nu (h+1)^\nu C_h(p)$. We therefore get

$$\begin{aligned}\left|\frac{C_h(p_\nu)}{C_l(p_\nu)} - r\right| &= \frac{(h+1)^\nu}{(l+1)^\nu} \cdot \left|\frac{C_h(p)}{C_l(p)} - r_0\right| \\ &< \frac{(h+1)^\nu}{(l+1)^\nu} \cdot \frac{\varepsilon}{2} \cdot \left(\frac{h+1}{l+1}\right)^{-\nu} = \frac{\varepsilon}{2}.\end{aligned}$$

Hence we can use exactly the same argument as before where instead of $p(X)$ we use the polynomial $p_\nu(X)$.

Finally, let $l > h \geq 1$ and let $\varepsilon' > 0$. We have shown that for all $r \in \mathbb{R}^+$ and

$$\varepsilon := \min\left(\frac{r}{2}, \frac{\varepsilon' r^2}{2 + \varepsilon' r}\right) > 0 \tag{22}$$

there is an integer n such that

$$\left|\frac{s_q(n^l)}{s_q(n^h)} - r\right| < \varepsilon.$$

Note that by (22), we have $r - \varepsilon > 0$. When we distinguish the two cases corresponding to the minimum in (22) we see that the same integer n also verifies

$$\left|\frac{s_q(n^h)}{s_q(n^l)} - \frac{1}{r}\right| = \left|\frac{r - \frac{s_q(n^l)}{s_q(n^h)}}{r\ \frac{s_q(n^l)}{s_q(n^h)}}\right| < \frac{\varepsilon}{r(r-\varepsilon)} \leq \frac{\varepsilon'}{2} < \varepsilon'.$$

This completes the proof of Theorem 1.1 in the case of $P_1(X) = X^h$, $P_2(X) = X^l$ with $h \neq l$.

2.5. **The case of general polynomials.** The general case of polynomials $P_1(X), P_2(X) \in \mathbb{Z}[X]$ with $P_1(\mathbb{N}), P_2(\mathbb{N}) \subset \mathbb{N}$ follows rather directly from the discussion for monomials. To begin with, there exists $n_0 = n_0(P_1, P_2)$ such that both $P_1(n + n_0)$ and $P_2(n + n_0)$ only have positive coefficients. We can therefore assume, without loss of generality, that both $P_1(X)$ and $P_2(X)$ have positive coefficients which are bounded by some constant only depending on P_1 and P_2. We construct $p(X)$ as before with the monomials X^h, X^l in place of $P_1(X)$, $P_2(X)$. We claim that the approach with $p(X)$ works as good as for $P_1(X)$, $P_2(X)$, provided $k \geq k(P_1, P_2)$ is sufficiently large.

Let $t \geq 1$ and consider

$$P(X) = \sum_{j=0}^{t} c_j X^j, \qquad c_j > 0, \quad 0 \leq j \leq t.$$

It is sufficient to show that $P(p(X))$ has the same sign structure in its expansion as $p(X)^t$ provided that $k = k(P)$ is sufficiently large. First, we know that our construction fills up completely the gaps between $Q_{i+1}(p(X)^t)$ and $Q_i(p(X)^t)$ for any sufficiently large (fixed) k. Moreover, recall that we have shown that for $1 \le j \le t$ and $0 \le i \le j$, the total weight attached to each power X^m with $m \in Q_i(p(X)^j)$ (resp. $G_i(p(X)^j)$) is at least q^{jk} (resp. is $O_{a,j}(q^{(j-1)k})$).

Now, the relations (7), (10) and a comparison of the interval bounds imply that for all i, j and v with $0 \le i \le t-1$, $1 \le j \le t-1$ and $0 \le v \le j$,

$$G_i(p(X)^t) \cap Q_v(p(X)^j) = \emptyset.$$

This means, that $P(p(X))$ has at least the same number of powers with negative coefficients as $p(X)^t$. On the other hand, if

$$Q_i(p(X)^t) \cap G_v(p(X)^j) \neq \emptyset$$

then, as the weight associated to elements of $Q_i(p(X)^t)$ is dominant, we can find a sufficiently large k such that the coefficients to powers X^m for $m \in \bigcup_{0 \le i \le t} Q_i(p(X))$ are positive. This shows, in particular, that for sufficiently large k the number of negative coefficients in the expansions of $P_1(p(X))$ (resp. $P_2(p(X))$) is $C_h(p)$ (resp. $C_l(p)$) and the same proof as before can be applied.

This completes the proof of Theorem 1.1. □

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