

NON-NORMAL NUMBERS WITH RESPECT TO MARKOV PARTITIONS

MANFRED G. MADRITSCH

ABSTRACT. We call a real number normal if for any block of digits the asymptotic frequency of this block in the N -adic expansion equals the expected one. In the present paper we consider non-normal numbers and, in particular, essentially and extremely non-normal numbers. We call a real number essentially non-normal if for each single digit there exists no asymptotic frequency of its occurrence. Furthermore we call a real number extremely non-normal if all possible probability vectors are accumulation points of the sequence of frequency vectors. Our aim now is to extend and generalize these results to Markov partitions.

1. INTRODUCTION

Let $N \geq 2$ be an integer, called the base, and $\mathcal{D} := \{0, 1, \dots, N-1\}$, called the set of digits. Then for every $x \in [0, 1)$ we denote by

$$x = \sum_{h=1}^{\infty} d_h(x) N^{-h},$$

where $d_h(x) \in \mathcal{D}$ for all $h \geq 1$, the unique non-terminating N -ary expansion of x . For every positive integer n and a block of digits $\mathbf{b} = b_1 \dots b_k \in \mathcal{D}^k$ we write

$$\Pi(x, \mathbf{b}, n) := \frac{|\{0 \leq i < n : d_{i+1}(x) = b_1, \dots, d_{i+k}(x) = b_k\}|}{n}$$

for the frequency of the block \mathbf{b} among the first n digits of the N -ary expansion of x . Furthermore let

$$\Pi_k(x, n) := (\Pi(x, \mathbf{b}, n))_{\mathbf{b} \in \mathcal{D}^k}$$

be the vector of frequencies of all blocks \mathbf{b} of length k .

Now we call a number k -normal if for every block $\mathbf{b} \in \mathcal{D}^k$ of digits of length k , the limit of the frequency $\Pi(x, \mathbf{b}, n)$ exists and equals N^{-k} . A number is called normal with respect to base N if it is k -normal for all $k \geq 1$. Furthermore a number is called absolutely normal if it is normal to any base $N \geq 2$.

On the one hand it is a classical result due to Borel [5] that Lebesgue almost all numbers are absolutely normal. So the set of normal numbers is large from a measure theoretical viewpoint.

On the other hand it suffices for a number to be not normal if the limit of the frequency vector is not the uniform one. First results concerning the Hausdorff dimension or the Baire category of non-normal numbers were obtained by Šalát [16] and Volkmann [19]. Stronger variants of non-normal numbers were of recent interest. In particular, Albeverio *et al.* [1, 2] considered the fractal structure of essentially non-normal numbers and their variants. The theory of multifractal divergence points lead to the investigation of extremely non-normal numbers by Olsen [11, 12] and Olsen and Winter [14]. The important result for our considerations is that both essentially and extremely non-normal numbers are large from a topological point of view. The present paper focuses on this dichotomy of the non-normal numbers: on the one hand they are a set of measure zero and on the other hand they are residual.

Before we get into the statements and their proofs we want to define essentially and extremely non-normal numbers. We call a number essentially non-normal if every single digit has no limit

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frequency, *i.e.* for every $b \in \mathcal{D}$ we have that

$$\lim_{n \rightarrow \infty} \Pi(x, b, n)$$

does not exist. Let \mathbb{L} be the set of essentially non-normal numbers in base N . Then Albeverio *et al.* [1, 2] could prove amongst other results the following

Theorem ([1, 2, Theorem 1]). *The set \mathbb{L} is residual, *i.e.* $[0, 1] \setminus \mathbb{L}$ is of the first category. In particular, the set \mathbb{L} is of the second category.*

Another class of non-normal numbers are extremely non-normal numbers. Let S_k be the set of shift invariant probability vectors, *i.e.*

$$S_k := \left\{ (p_{\mathbf{b}})_{\mathbf{b} \in \mathcal{D}^k} : p_{\mathbf{b}} \geq 0, \sum_{\mathbf{b} \in \mathcal{D}^k} p_{\mathbf{b}} = 1, \sum_{d \in \mathcal{D}} p_{d\mathbf{b}} = \sum_{d \in \mathcal{D}} p_{\mathbf{b}d} \text{ for all } \mathbf{b} \in \mathcal{D}^{k-1} \right\}.$$

Then we call a number $x \in [0, 1]$ extremely non- k -normal (with respect to base N) if each shift invariant probability vector $\mathbf{p} \in S_k$ occurs as accumulation point of the sequence $(\Pi_k(x, n))_n$. Furthermore we call a number $x \in [0, 1]$ extremely non-normal (with respect to N) if it is extremely non- k -normal (with respect to N) for every $k \geq 1$. We denote by $\mathbb{E}_{k,N}$ and \mathbb{E}_N the sets of extremely non- k -normal and extremely non-normal numbers, respectively. Finally let \mathbb{E} be the set of extremely non-normal numbers to all bases $N \geq 2$, *i.e.*

$$\mathbb{E}_{k,N} = \{x \in [0, 1] : \text{each } \mathbf{p} \in S_k \text{ is an accumulation point of the sequence } (\Pi_k(x, n))_n\},$$

$$\mathbb{E}_N = \bigcap_{k \geq 1} \mathbb{E}_{k,N}, \quad \mathbb{E} = \bigcap_{N \geq 2} \mathbb{E}_N.$$

Olsen [13] was able to prove that the set \mathbb{E} is big in the topological sense.

Theorem ([13, Theorem 1]). *The set \mathbb{E} is residual, *i.e.* $[0, 1] \setminus \mathbb{E}$ is of the first category. In particular, the set \mathbb{E} is of the second category.*

This result was generalized to iterated function systems by Baek and Olsen [3]. This result stays in connection with our Theorem 2.4. However, on the one hand we use a direct approach for the proof, whereas Baek and Olsen map the iterated function system to the N -ary number system. On the other hand we also consider Cesàro variants of the extremely normal numbers. Furthermore number systems with infinite set of digits like the continued fraction expansion and to Lüroth expansion were considered by Olsen [10] and Šalát [15] respectively. Finally Šalát [17] considered the Hausdorff dimension of sets with digital restrictions with respect to the Cantor series expansion.

We now want to introduce the Cesàro variants of extremely non-normal numbers. The main idea is that if the limit $\Pi(x, b, n)$ does not exist, the Cesàro limit might. In particular, for $i \in \mathcal{D}$ let

$$\Pi^{(1)}(x, i, n) = \Pi(x, i, n),$$

and for $r \geq 2$, let

$$\Pi^{(r)}(x, i, n) := \frac{\sum_{j=1}^n \Pi^{(r-1)}(x, i, j)}{n}$$

be the r th iterated Cesàro average. As above let

$$\mathbf{\Pi}^{(r)}(x, n) = (\Pi^{(r)}(x, 0, n), \dots, \Pi^{(r)}(x, N-1, n))$$

denote the vector of r th iterated Cesàro averages. Then Hyde *et al.* [8] considered the Cesàro average of the frequencies and were able to show the following

Theorem ([8, Theorem 1.1]). *For all integers $N \geq 2$ and $r \geq 1$ the set*

$$\{x \in [0, 1] : \text{the set of accumulation points of } (\mathbf{\Pi}^{(r)}(x, n))_{n=1}^{\infty} \text{ equals } S_1\}$$

is residual.

2. DEFINITIONS AND STATEMENT OF RESULT

The main goal of this paper is to generalize and extend the above mentioned results to Markov partitions. In our definitions regarding symbolic dynamical systems we mainly follow Chapter 6 of Lind and Marcus [9].

We start with the definition of a dynamical system. Let M be a compact metric space and $\phi : M \rightarrow M$ be a continuous map. Then we call the pair (M, ϕ) a dynamical system.

The second ingredient is the definition of a topological partition. Let M be a metric space and let $\mathcal{P} = \{P_0, \dots, P_{N-1}\}$ be a finite collection of disjoint open sets. Then we call \mathcal{P} a topological partition (of M) if M is the union of the closures $\overline{P_i}$ for $i = 0, \dots, N-1$, *i.e.*

$$M = \overline{P_0} \cup \dots \cup \overline{P_{N-1}}.$$

Suppose now that a dynamical system (M, ϕ) and a topological partition $\mathcal{P} = \{P_0, \dots, P_{N-1}\}$ of M are given. We want to consider the symbolic dynamical system behind. Therefore let $\Sigma = \{0, \dots, N-1\}$ be the alphabet corresponding to the topological partition \mathcal{P} . Furthermore define

$$\Sigma^k = \{0, \dots, N-1\}^k, \quad \Sigma^* = \bigcup_{k \geq 1} \Sigma^k, \quad \text{and} \quad \Sigma^{\mathbb{N}} = \{0, \dots, N-1\}^{\mathbb{N}}$$

to be the set of words of length k , the set of finite and the set of infinite words over Σ , respectively. For an infinite word $\omega = a_1 a_2 a_3 \dots \in \Sigma^{\mathbb{N}}$ and a positive integer n , let $\omega|n = a_1 a_2 \dots a_n$ denote the truncation of ω to the n -th place. Finally for $\omega \in \Sigma^*$ we denote by $[\omega]$ the cylinder set of all infinite words starting with the same letters as ω , *i.e.*

$$[\omega] := \{\gamma \in \Sigma^{\mathbb{N}} : \gamma| |\omega| = \omega\}.$$

Now we want to describe the shift space that is generated by our Markov partition. Therefore we call $\omega = a_1 a_2 a_3 \dots \in \Sigma^{\mathbb{N}}$ allowed for (\mathcal{P}, ϕ) if

$$\bigcap_{k=1}^n \phi^{-k}(P_{a_k}) \neq \emptyset.$$

Let $\mathcal{L}_{\mathcal{P}, \phi}$ be the set of allowed words. Then $\mathcal{L}_{\mathcal{P}, \phi}$ is a language and there is a unique shift space $X_{\mathcal{P}, \phi} \subseteq \Sigma^{\mathbb{N}}$, whose language is $\mathcal{L}_{\mathcal{P}, \phi}$. We call $X_{\mathcal{P}, \phi} \subseteq \Sigma^{\mathbb{N}}$ the one-sided symbolic dynamical system corresponding to (\mathcal{P}, ϕ) . Finally for each $\omega = a_1 a_2 a_3 \dots \in X_{\mathcal{P}, \phi}$ and $n \geq 0$ we denote by $D_n(\omega)$ the cylinder set of order n corresponding to ω in M , *i.e.*,

$$D_n(\omega) := \bigcap_{k=0}^n \phi^{-k}(P_{a_k}) \subseteq M.$$

Now we can state the definition of a Markov partition.

Definition 2.1. Let (M, ϕ) be a dynamical system and $\mathcal{P} = \{P_0, \dots, P_{N-1}\}$ be a topological partition of M . Then we call \mathcal{P} a Markov partition if the generated shift space $X_{\mathcal{P}, \phi}$ is of finite type and for every $\omega \in X_{\mathcal{P}, \phi}$ the intersection $\bigcap_{n=0}^{\infty} \overline{D_n(\omega)}$ consists of exactly one point.

After providing all the ingredients necessary for the statement of our result we want to link the introduced concept of Markov partitions with the N -ary representations of Section 1 and with matrix number systems (cf. Gröchenig and Haas [7]).

Example 1. Let $M = \mathbb{R}/\mathbb{Z}$ be the circle and $\phi : M \rightarrow M$ be defined by $\phi(x) = Nx \pmod{1}$. We divide M into N subintervals P_0, \dots, P_{N-1} of the form $P_i = (i/N, (i+1)/N)$ and let $\Sigma = \{0, \dots, N-1\}$. Then the underlying system is the N -ary representation. Furthermore it is easy to verify that the language $\mathcal{L}_{\mathcal{P}, \phi}(x)$ is the set of all words over Σ , so that the one-sided symbolic dynamical system $X_{\mathcal{P}, \phi}$ is the full one-sided N -shift $\Sigma^{\mathbb{N}}$.

Example 2. Let $A \in \mathbb{Z}^{n \times n}$ be an expanding matrix and let $\mathcal{D} \subset \mathbb{Z}^n$ be a complete set of residues modulo $A\mathbb{Z}^n$. Then there exists a unique compact set M such that

$$M = \bigcup_{d \in \mathcal{D}} A^{-1}(M + d).$$

The sets

$$P_d := (A^{-1}(K + d))^\circ$$

together with $\phi : Ax \bmod AZ^n$ form a Markov partition of M . It is again easy to verify that the corresponding language is the set of all words over \mathcal{D} , which can be mapped to $\{0, \dots, N-1\}$ for some integer N .

We note that Example 2 also contains the cases of canonical number systems as a special case. For further information on the different dynamical aspects we refer the interested reader to the survey of Barat *et al.* [4] and the references therein.

In order to extend the definition of normal and thus non-normal numbers to M we need the map $\pi_{\mathcal{P},\phi} : X_{\mathcal{P},\phi} \rightarrow M$ defined by

$$\{\pi_{\mathcal{P},\phi}(\omega)\} = \bigcap_{n=0}^{\infty} \overline{D_n(\omega)}.$$

By the definition of a Markov partition we have that every $\omega \in X_{\mathcal{P},\phi}$ maps to a unique element $x \in M$. However, the converse need not be true. In particular, let us consider Example 1 with $N = 10$ (the decimal expansion in the unit interval). Then on the one hand every expansion is mapped to a unique real number. On the other hand the expansions $0.99999\dots$ and $1.00000\dots$ correspond to the same element. Similarly we get that $0.39999\dots = 0.40000\dots$. However, one observes that these ambiguities originate from the intersections $\overline{P_i} \cap \overline{P_j}$ for $i \neq j$. Thus we concentrate on the inner points, which somehow correspond to the irrational numbers in the above case of the decimal expansion. Let

$$U = \bigcup_{i=0}^{N-1} P_i,$$

which is an open and dense ($\overline{U} = M$) set. Then for each $n \geq 1$ the set

$$U_n = \bigcap_{k=0}^{n-1} \phi^{-k}(U),$$

is open and dense in M . Thus by the Baire Category Theorem, the set

$$(2.1) \quad U_\infty = \bigcap_{n=0}^{\infty} U_n$$

is dense. Since $M \setminus U_\infty$ is the countable union of nowhere dense sets it suffices to show that a set is residual in U_∞ in order to show that it is residual in M . Furthermore for $x \in U_\infty$ we may call ω **the** symbolic expansion of x if $\pi_{\mathcal{P},\phi}(\omega) = x$. Thus in the following we will silently suppose that $x \in U_\infty$.

After defining the environment we want to pull over the definitions of normal and non-normal numbers to the symbolic dynamical system. To this end let $\mathbf{b} \in \Sigma^k$ be a block of letters of length k and $\omega = a_1 a_2 a_3 \dots \in X_{\mathcal{P},\phi}$ be the symbolic representation of an element. Then we write

$$P(\omega, \mathbf{b}, n) = |\{0 \leq i < n : a_{i+1} = b_1, \dots, a_{i+k} = b_k\}|$$

for the frequency of the block \mathbf{b} among the first n letters of ω . In the same manner as above let

$$P_k(\omega, n) = (P(\omega, \mathbf{b}), n)_{\mathbf{b} \in \Sigma^k}$$

be the vector of all frequencies of blocks \mathbf{b} of length k among the first n letters of ω .

In order to properly define normal numbers we need a probability measure on M . Therefore let \mathfrak{B} be the σ -algebra generated by the cylinder sets of M and μ be a probability measure defined on \mathfrak{B} . If $\phi : M \rightarrow M$ is measure preserving, *i.e.* $\mu(\phi^{-1}(A)) = \mu(A)$ for all $A \in \mathfrak{B}$, then we call the quadruple $(M, \phi, \mathfrak{B}, \mu)$ a measure-theoretic dynamical system. Moreover we call it ergodic if ϕ is ergodic, *i.e.* $\mu(\phi^{-1}A \Delta A) = 0$ implies that either $\mu(A) = 1$ or $\mu(A) = 0$.

A number $\omega \in X_{\mathcal{P},\phi}$ is called k -normal (with respect to $(M, \phi, \mathfrak{B}, \mu)$) if for all blocks $\mathbf{b} \in \Sigma^k$ of length k we have

$$\lim_{n \rightarrow \infty} P_k(\omega, n) = \mu(D_k(\mathbf{b}))$$

Furthermore, a number $\omega \in X_{\mathcal{P},\phi}$ is called normal (with respect to $(M, \phi, \mathfrak{B}, \mu)$) if it is k -normal for every $k \geq 1$. An application of Birkhoff's ergodic theorem yields for μ being ergodic that almost all numbers $\omega \in X_{\mathcal{P},\phi}$ are normal (cf. Chapter 3.1.2 of [6]).

We note that we equivalently could have defined the measure-theoretic dynamical system with respect to $X_{\mathcal{P},\phi}$ instead of M . However, since the definition of essentially and extremely non-normal numbers does not depend on this, we will not consider this in the following.

As already mentioned above the aim of the present paper is to show that the non-normal numbers are a large set in the topological sense. In particular, we want to generalize the theorems from above by Albeverio *et al.* [2] and Olsen [13] to this new setting. Thus we call a number $\omega \in X_{\mathcal{P},\phi}$ essentially non-normal if for all $i \in \Sigma$ the limit

$$\lim_{n \rightarrow \infty} P(\omega, i, n)$$

does not exist. By abuse of notation we denote by \mathbb{L} the set of essentially non-normal numbers in M . Then our first result is the following

Theorem 2.1. *Let $\mathcal{P} = \{P_0, \dots, P_{N-1}\}$ be a one-sided Markov partition for (M, ϕ) . Suppose that $X_{\mathcal{P},\phi}$ is the one-sided full-shift. Then the set of essentially non-normal numbers is residual.*

Remark 2.2. The requirement that $X_{\mathcal{P},\phi}$ is the full-shift is an artifact of the used construction. In particular, we suppose that a similar result can be shown for any shift of finite type fulfilling some mild conditions in order to exclude some trivial cases. For example, we want to exclude the case of the shift over the alphabet $\{0, 1\}$ with forbidden words 00 and 11.

A different concept of non-normal numbers are those being arbitrarily close to any given configuration. In particular, we want to generalize the idea of extremely non-normal numbers and their Cesàro variants to the setting of Markov partitions. Following Olsen's paper [13] we start by defining the simplex of all probability vectors Δ_k by

$$\Delta_k = \left\{ (p_{\mathbf{i}})_{\mathbf{i} \in \Sigma^k} : p_{\mathbf{i}} \geq 0, \sum_{\mathbf{i} \in \Sigma^k} p_{\mathbf{i}} = 1 \right\}.$$

Let $\|\cdot\|_1$ denote the 1-norm then $(\Delta_k, \|\cdot\|_1)$ is a metric space. On the one hand we clearly have that any vector $P_k(\omega, n)$ of frequencies of blocks of digits of length k belongs to Δ_k . On the other hand we want to quantify the non-normality of a number by considering the extend to which the sequence $(P_k(\omega, n))_n$ fills up the simplex Δ_k . Following the arguments of Volkmann [18] or Olsen [10, 13] we get that the sequences $(P_k(\omega, n))_n$ can only fill up an essential part of Δ_k for any ω and k . In particular, let us consider all possible ways a block of length 2, such as 28, can give rise to one of length 3. Thus we get that

$$(2.2) \quad \left| \sum_{i \in \Sigma} P(\omega, i28, n) - \sum_{i \in \Sigma} P(\omega, 28i, n) \right| \leq \frac{1}{n}$$

for all ω . This implies, that for each ω , all but finitely many points in the sequence $(P_3(\omega, n))_n$ will be very close to the subsimplex

$$\Delta_3 \cap \left\{ (p_{\mathbf{i}})_{\mathbf{i} \in \Sigma^3} : \sum_{i \in \Sigma} p_{i28} = \sum_{i \in \Sigma} p_{28i} \right\}.$$

Thus we get as Olsen [13] that the sequence $(P_k(\omega, n))_n$ does not fill up a significant part of Δ_k . In particular, the simplex Δ_k is not the "correct" object to consider. Rather we need to consider the subsimplex of shift invariant probability vectors S_k , *i.e.*

$$S_k := \left\{ (p_{\mathbf{i}})_{\mathbf{i} \in \Sigma^k} : p_{\mathbf{i}} \geq 0, \sum_{\mathbf{i} \in \Sigma^k} p_{\mathbf{i}} = 1, \sum_{i \in \Sigma} p_{i\mathbf{i}} = \sum_{i \in \Sigma} p_{\mathbf{i}i} \text{ for all } \mathbf{i} \in \Sigma^{k-1} \right\}.$$

Now we define the second ingredient for extremely non-normal numbers, namely the accumulation points of the frequency vectors. In particular, let $A_k(\omega)$ be the set of accumulation points

of the sequence $(P_k(\omega, n))_n$ with respect to $\|\cdot\|_1$, *i.e.* for $\omega \in X_{\mathcal{P}, \phi}$ we set

$$A_k(\omega) := \{\mathbf{p} \in \Delta_k : \mathbf{p} \text{ is an accumulation point of } (P_k(\omega, n))_n\}.$$

Our first step is to show, that the sub-simplex S_k is really the right object of investigation. Therefore we have the following

Theorem 2.3. *Let $\omega \in X_{\mathcal{P}, \phi}$. Then*

$$A_k(\omega) \subset S_k.$$

Proof. We prove this by showing that for each accumulation point also its shifts must lie in $A_k(\omega)$. Let $\mathbf{p} = (p_i)_{i \in \Sigma^k}$ be an accumulation point of the sequence $(P_k(\omega, n))_n$ with respect to the 1-norm. Then there exists an increasing sequence $(n_m)_m$ of positive integers such that

$$\|(P_k(\omega, n_m))_m - \mathbf{p}\|_1 \rightarrow 0.$$

Now we use the idea of (2.2) in order to consider all the possible ways a block of length $k-1$ can be extended to one of length k . Thus

$$\begin{aligned} \left| \sum_{i \in \Sigma} p_{ii} - \sum_{i \in \Sigma} p_{ii} \right| &\leq \left| \sum_{i \in \Sigma} p_{ii} - P_k(\omega, \mathbf{ii}, n_m) \right| + \left| \sum_{i \in \Sigma} P_k(\omega, \mathbf{ii}, n_m) - \sum_{i \in \Sigma} P_k(\omega, \mathbf{i}i, n_m) \right| \\ &\quad + \left| \sum_{i \in \Sigma} P_k(\omega, \mathbf{i}i, n_m) - \sum_{i \in \Sigma} p_{ii} \right| \\ &\leq \|(P_k(\omega, n_m))_m - \mathbf{p}\|_1 + \frac{1}{n_m} + \|(P_k(\omega, n_m))_m - \mathbf{p}\|_1 \rightarrow 0, \end{aligned}$$

which implies that $\sum_{i \in \Sigma} p_{ii} = \sum_{i \in \Sigma} p_{ii}$ for all $\mathbf{i} \in \Sigma^{k-1}$. \square

We call an infinite word $\omega \in X_{\mathcal{P}, \phi}$ extremely non- k -normal if the set of accumulation points of the sequence $(P_k(\omega, n))_n$ (with respect to $\|\cdot\|_1$) equals S_k , *i.e.* $A_k(\omega) = S_k$. Furthermore we call a number extremely non-normal if it is extremely non- k -normal for all $k \geq 1$.

As above we also want to extend this notion to the Cesàro averages of the frequencies. To this end for a fixed block $b_1 \dots b_k \in \Sigma^k$ let

$$P^{(1)}(\omega, \mathbf{b}, n) = P(\omega, \mathbf{b}, n).$$

For $r \geq 2$ we recursively define

$$P^{(r)}(\omega, \mathbf{b}, n) = \frac{\sum_{j=1}^n P^{(r-1)}(\omega, \mathbf{b}, j)}{n}$$

to be the r th iterated Cesàro average of the frequency of the block of digits \mathbf{b} under the first n digits. Furthermore we define by

$$P_k^{(r)}(\omega, n) := \left(P^{(r)}(\omega, \mathbf{b}, n) \right)_{\mathbf{b} \in \Sigma^k}$$

the vector of r th iterated Cesàro averages. As above we are interested in the accumulation points. Thus similar to above let $A_k^{(r)}(\omega)$ denote the set of accumulation points of the sequence $(P_k^{(r)}(\omega, n))_n$ with respect to $\|\cdot\|_1$, *i.e.*

$$A_k^{(r)}(\omega) := \left\{ \mathbf{p} \in \Delta_k : \mathbf{p} \text{ is an accumulation point of } (P_k^{(r)}(\omega, n))_n \right\}.$$

We will denote the set of extremely non- k -normal numbers of M by $\mathbb{E}_k^{(1)}$. Similarly for $r \geq 1$ and $k \geq 1$ we denote by $\mathbb{E}_k^{(r)}$ the set of r th iterated Cesàro extremely non- k -normal numbers of M . Furthermore for $r \geq 1$ we denote by $\mathbb{E}^{(r)}$ the set of r th iterated Cesàro extremely non-normal numbers and by \mathbb{E} the set of completely Cesàro extremely non-normal numbers, *i.e.*

$$\mathbb{E}^{(r)} = \bigcap_k \mathbb{E}_k^{(r)} \quad \text{and} \quad \mathbb{E}^{(r)} = \bigcap_r \mathbb{E}^{(r)} = \bigcap_{r,k} \mathbb{E}_k^{(r)}.$$

Then our result is the following

Theorem 2.4. *Let k , r and N be positive integers. Furthermore let $\mathcal{P} = \{P_0, \dots, P_{N-1}\}$ be a one-sided Markov partition for (M, ϕ) . Suppose that $X_{\mathcal{P}, \phi}$ is the one-sided full-shift. Then the set $\mathbb{E}_k^{(r)}$ is residual.*

Since the set of non-normal numbers is a countable intersection of sets $\mathbb{E}_k^{(r)}$ we get the following generalization of the result by Hyde *et al.* [8].

Corollary 2.5. *Let N be a positive integer and $\mathcal{P} = \{P_0, \dots, P_{N-1}\}$ be a one-sided Markov partition for (M, ϕ) . Suppose that $X_{\mathcal{P}, \phi}$ is the one-sided full-shift. Then the sets $\mathbb{E}^{(r)}$ and \mathbb{E} are residual.*

3. PROOF OF THEOREM 2.1

We borrow the method of Albeverio *et al.* [2] in order to construct a set of infinite words such that no limiting frequency of a single digit exists. In particular, this means that this is a subset of the set of essentially non-normal numbers, which is already residual.

For $n \geq 1$ let γ_n be the word

$$(3.1) \quad \gamma_n = \underbrace{00 \dots 0}_n \underbrace{11 \dots 1}_n \dots \underbrace{(N-1)(N-1) \dots (N-1)}_n$$

and C_n be the set

$$(3.2) \quad C_n = \bigcup_{\alpha \in \Sigma^n} [\alpha \gamma_n].$$

We fix $\omega \in C_n$ and a digit $i \in \Sigma$. Then we show that C_n has the desired properties. In particular, we compare the number of occurrences of i in ω just after the block of i s in γ_n and after the final block of $(N-1)$ s. Thus for $i \in \{0, \dots, N-2\}$ we set $k'_n(i) = n(i+2)$ and $k''_n(i) = n(N+1)$. Then we get that

$$P(\omega, i, k'_n(i)) = \frac{n + P(\omega, i, n)}{n(i+2)}, \quad P(\omega, i, k''_n(i)) = \frac{n + P(\omega, i, n)}{n(N+1)}.$$

One easily checks that

$$|P(\omega, i, k'_n(i)) - P(\omega, i, k''_n(i))| \geq \frac{1}{(i+2)(N+1)} \geq \frac{1}{(N+1)^2}.$$

For the case of $i = N-1$ we set $k'_n(i) = nN$ and $k''_n(i) = n(N+1)$. Then we get that

$$|P(\omega, i, k'_n(i)) - P(\omega, i, k''_n(i))| = \frac{nN - P(x, i, n)}{N(N+1)n} \geq \frac{1}{(N+1)^2}.$$

Since ω was arbitrary we get for every $i \in \Sigma$ two sub-sequences of positions $k'_n(i)$ and $k''_n(i)$ such that their respective number of occurrences of the digit i tends to different limits.

In the next step we carry this construction over to the set U_∞ defined in (2.1), which is sufficient for the proof. For $n \geq 1$ we set

$$I_n = \bigcup_{\alpha \in \Sigma^n} D_{(N+1)n}(\alpha \gamma_n).$$

We note that these sets are the ‘‘inner points’’ of the sets C_n . Any point in I_n has a corresponding representation in C_n . Furthermore we set

$$F_n = \bigcup_{k=n}^{\infty} I_k \quad \text{and} \quad L = \bigcap_{n=1}^{\infty} F_n.$$

In order to show that L is residual in U_∞ it suffices to show that every F_n is open and dense.

- **F_n is open.** F_n is the union of open sets I_k and therefore open.
- **F_n is dense.** Let $x \in U_\infty$ and $\delta > 0$. Then we have to show that there exists $y \in F_n \cap B(x, \delta)$. Let $\omega \in X$ be the expansion corresponding to x , *i.e.* $\pi(\omega) = x$. Since $\text{diam } \overline{D_k(\omega)} \rightarrow 0$ and $x \in D_k(\omega)$ we may choose $k \geq n$ such that $D_k(\omega) \subset B(x, \delta)$. We set $\sigma = \omega \upharpoonright k$ and chose $y \in D_{(N+1)k}(\sigma \gamma_k)$. Then since $D_{(N+1)k}(\sigma \gamma_k) \subset I_k \subset F_n$ and $D_{(N+1)k}(\sigma \gamma_k) \subset D_k(\omega) \subset B(x, \delta)$ we have that $y \in F_n \cap B(x, \delta)$.

Since L is the countable union of open and dense sets we get that L is residual in U_∞ .

Finally we note that if $x \in L$, then $x \in F_n$, $\forall n \in \mathbb{N}$. Therefore there exists an increasing sequence of integers $\{n_m\}$ such that $x \in I_{n_m}$ for all $m \in \mathbb{N}$. Thus for any fixed $x \in L$ and every $i \in \Sigma$ one can choose two sequences $k'_{n_m}(i)$ and $k''_{n_m}(i)$ such that

$$|\mathbb{P}(\omega, i, k'_n(i)) - \mathbb{P}(\omega, i, k''_n(i))| \geq \frac{1}{(N+1)^2},$$

where ω is the symbolic expansion of x . Hence, $L \subset \mathbb{L}$, which proves Theorem 2.1.

4. PROOF OF THEOREM 2.4

Now we draw our attention to the case of extremely non-normal numbers and their Cesàro variants. Let $k \in \mathbb{N}$ and $\mathbf{q} \in S_k$ be fixed throughout the rest of this section. For $n \geq 1$ we define a set Z_n which we will use in order to “measure” the distance of the number of occurrences and \mathbf{q} , *i.e.*

$$Z_n = Z_n(\mathbf{q}, k) = \left\{ \omega \in \bigcup_{\ell \geq kn} \Sigma^\ell \mid \|\mathbb{P}_k(\omega) - \mathbf{q}\|_1 \leq \frac{1}{n} \right\}.$$

The main idea consists now in the construction of a word having the desired frequencies. In particular, for a given word ω we want to show that we can add sufficiently many copies of a word from Z_n to get a word whose frequency vector is sufficiently near to \mathbf{q} . Therefore we first need, that Z_n is not empty.

Lemma 4.1 ([10, Lemma 2.4]). *For all $n \geq 1$, $\mathbf{q} \in S_k$ and $k \in \mathbb{N}$ we have $Z_n(\mathbf{q}, k) \neq \emptyset$.*

Now we consider how many copies of elements in Z_n we have to add in order to get the desired properties.

Lemma 4.2. *Let $\mathbf{q} \in S_k$ and $n, t \in \mathbb{N}$ be positive integers. Furthermore let $\omega = \omega_1 \dots \omega_t \in \Sigma^t$ be a word of length t . Then, for any $\gamma \in Z_n(\mathbf{q}, k)$ and any*

$$(4.1) \quad \ell \geq L := t + |\gamma| \max\left(n, \frac{t}{k}\right)$$

we get that

$$\|\mathbb{P}_k(\omega\gamma^*|\ell) - \mathbf{q}\| \leq \frac{6}{n}.$$

Proof. We set $s := |\gamma|$ and

$$\sigma = \omega\gamma^*|\ell.$$

Furthermore we set q and $0 \leq r < s$ such that $m = t + qs + r$. For a fixed block \mathbf{i} an occurrence can happen in ω , in γ , somewhere in between or at the end. Thus for every $\mathbf{i} \in \Sigma^k$ we clearly have that

$$\frac{qs}{\ell} \mathbb{P}(\gamma, \mathbf{i}) \leq \mathbb{P}(\sigma, \mathbf{i}) \leq \frac{qs}{\ell} \mathbb{P}(\gamma, \mathbf{i}) + \frac{t + q(k-1) + r}{\ell}.$$

Now we concentrate on the occurrences in multiples of γ and show that we may neglect those outside of γ , *i.e.*,

$$\|\mathbb{P}_k(\sigma) - \mathbf{q}\| \leq \left\| \mathbb{P}_k(\sigma) - \frac{qs}{\ell} \mathbb{P}_k(\gamma) \right\| + \left\| \frac{qs}{\ell} \mathbb{P}_k(\gamma) - \mathbf{q} \right\|.$$

We will estimate both parts separately. For the first one we get that

$$\begin{aligned} \left\| \mathbb{P}_k(\sigma) - \frac{qs}{\ell} \mathbb{P}_k(\gamma) \right\| &= \sum_{\mathbf{i} \in \Sigma^k} \left| \mathbb{P}(\sigma, \mathbf{i}) - \frac{qs}{\ell} \mathbb{P}(\gamma, \mathbf{i}) \right| \leq \sum_{\mathbf{i} \in \Sigma^k} \frac{t + qk + s}{\ell} \\ &\leq N^k \frac{t + qk}{qnkN^k} + \frac{1}{q} = \frac{t}{qnk} + \frac{1}{n} + \frac{1}{q}. \end{aligned}$$

where we have used that $\ell \geq qs \geq qnkN^k$.

For the second part we get that

$$\begin{aligned} \left\| \frac{qs}{\ell} P_k(\gamma) - \mathbf{q} \right\| &\leq \left\| \frac{qs}{\ell} P_k(\gamma) - P_k(\gamma) \right\| + \|P_k(\gamma) - \mathbf{q}\| \\ &\leq qs \left| \frac{1}{\ell} - \frac{1}{qs} \right| + \frac{1}{n} \\ &\leq \frac{t}{\ell} + \frac{1}{n} \leq \frac{t}{qnkN^k} + \frac{1}{n}. \end{aligned}$$

Putting these together yields

$$\|P_k(\sigma) - \mathbf{q}\| \leq \frac{2}{n} + \frac{t}{qnk} + \frac{1}{q} + \frac{t}{qnkN^k}.$$

By our assumptions on the size of ℓ in (4.1) this proves the lemma. \square

As in the papers of Olsen [10, 13] our main idea is to construct a residual set $E \subset \mathbb{E}_k^{(r)}$. But before we start we want to ease up notation. To this end we recursively define the function $\varphi_1(x) = 2^x$ and $\varphi_m(x) = \varphi_1(\varphi_{m-1}(x))$ for $m \geq 2$. Furthermore we set $\mathbb{D} = (\mathbb{Q}^N \cap \Delta_N)$. Since \mathbb{D} is countable and dense in Δ_N we may concentrate on the probability vectors $\mathbf{q} \in \mathbb{D}$.

Now we say that a sequence $(\mathbf{x}_n)_n$ in \mathbb{R}^{N^k} has property P if for all $\mathbf{q} \in \mathbb{D}$, $m \in \mathbb{N}$, $i \in \mathbb{N}$, and $\varepsilon > 0$, there exists a $j \in \mathbb{N}$ satisfying:

- (1) $j \geq i$,
- (2) $j/2^j < \varepsilon$,
- (3) if $j < n < \varphi_m(j)$ then $\|\mathbf{x}_n - \mathbf{q}\| < \varepsilon$.

Then we define our set E to consist of all frequency vectors having property P , *i.e.*

$$E = \{x \in U_\infty : (P_k^{(1)}(x; n))_{n=1}^\infty \text{ has property } P\}.$$

We will proceed in three steps showing that

- (1) E is residual,
- (2) if $(P^{(r)}(x; n))_{n=1}^\infty$ has property P , then also $(P^{(r+1)}(x; n))_{n=1}^\infty$ has property P , and
- (3) $E \subseteq \mathbb{E}_k^{(r)}$.

Lemma 4.3. *The set E is residual.*

Proof. For fixed $h, m, i \in \mathbb{N}$ and $\mathbf{q} \in \mathbb{D}$, we say that a sequence $(\mathbf{x}_n)_n$ in \mathbb{R}^{N^k} has property $P_{h,m,\mathbf{q},i}$ if for every $\varepsilon > 1/h$, there exists $j \in \mathbb{N}$ satisfying:

- (1) $j \geq i$,
- (2) $j/2^j < \varepsilon$,
- (3) if $j < n < \varphi_m(2^j)$, then $\|\mathbf{x}_n - \mathbf{q}\| < \varepsilon$.

Now let $E_{h,m,\mathbf{q},i}$ be the set of all points whose frequency vector satisfies property $P_{h,m,\mathbf{q},i}$, *i.e.*

$$E_{h,m,\mathbf{q},i} := \left\{ x \in U_\infty : \left(P_k^{(1)}(x; n) \right)_{n=1}^\infty \text{ has property } P_{h,m,\mathbf{q},i} \right\}.$$

Obviously we have that

$$E = \bigcap_{h \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcap_{\mathbf{q} \in \mathbb{D}} \bigcap_{i \in \mathbb{N}} E_{h,m,\mathbf{q},i}.$$

Thus it remains to show, that $E_{h,m,\mathbf{q},i}$ is open and dense.

- (1) $E_{h,m,\mathbf{q},i}$ **is open.** Let $x \in E_{h,m,\mathbf{q},i}$, then there exists a $j \in \mathbb{N}$ such that $j \geq i$, $j/2^j < 1/h$, and if $j < n < \varphi_m(2^j)$, then

$$\left\| P_k^{(1)}(x; n) - \mathbf{q} \right\|_1 < 1/h.$$

Let $\omega \in X$ be such that $x = \pi(\omega)$ and set $t := \varphi_m(2^j)$. Since $D_t(\omega)$ is open, there exists a $\delta > 0$ such that the ball $B(x, \delta) \subseteq D_t(\omega)$. Furthermore, since all $y \in D_t(\omega)$ have their first $\varphi_m(2^j)$ digits the same as x , we get that

$$B(x, \delta) \subseteq D_t(\omega) \subseteq E_{h,m,\mathbf{q},i}.$$

(2) $E_{h,m,\mathbf{q},i}$ is dense. Let $x \in U_\infty$ and $\delta > 0$. We must find $y \in B(x, \delta) \cap E_{h,m,\mathbf{q},i}$.

Let $\omega \in X$ be such that $x = \pi(\omega)$. Since $\overline{D_t(\omega)} \rightarrow 0$ and $x \in D_t(\omega)$ there exists a t such that $D_t(\omega) \subset B(x, \delta)$. Let $\sigma = \omega|t$ be the first t digits of x .

Now, an application of Lemma 4.1 yields that there exists a finite word γ such that

$$\|\mathbf{P}_k(\gamma) - \mathbf{q}\| \leq \frac{1}{6h}.$$

Let $\varepsilon \geq \frac{1}{h}$ and L be as in the statement of Lemma 4.2. Then we choose j such that

$$\frac{j}{2^j} < \varepsilon \quad \text{and} \quad j \geq \max(L, i).$$

An application of Lemma 4.2 then gives us that

$$\|\mathbf{P}_k(\sigma\gamma^*|j) - \mathbf{q}\| \leq \frac{6}{n} = \frac{1}{h} \leq \varepsilon.$$

Thus we choose $y \in D_j(\sigma\gamma^*)$. Then on the one hand $y \in D_j(\sigma\gamma^*) \subset D_t(\omega) \subset B(x, \delta)$ and on the other hand $y \in D_j(\sigma\gamma^*) \subset E_{h,m,\mathbf{q},i}$

It follows that E is the countable intersection of open and dense sets and therefore E is residual in U_∞ . \square

Lemma 4.4. *Let $\omega \in X_{\mathcal{P},\phi}$. If $(\mathbf{P}_k^{(r)}(\omega, n))_{n=1}^\infty$ has property P , then also $(\mathbf{P}_k^{(r+1)}(\omega, n))_{n=1}^\infty$ has property P .*

This is Lemma 2.2 of [8]. However, the proof is short so we present it here for completeness.

Proof. Let $\omega \in X_{\mathcal{P},\phi}$ be such that $(\mathbf{P}_k^{(r)}(\omega, n))_{n=1}^\infty$ has property P , and fix $\varepsilon > 0$, $\mathbf{q} \in S_k$, $i \in \mathbb{N}$ and $m \in \mathbb{N}$. Since $(\mathbf{P}_k^{(r)}(\omega, n))_{n=1}^\infty$ has property P , there exists $j' \in \mathbb{N}$ with $j' \geq i$, $j'/2^{j'} < \varepsilon/3$, and such that for $j' < n < \varphi_{m+1}(2^{j'})$ we have that $\|\mathbf{P}_k^{(r)}(\omega, n) - \mathbf{q}\| < \varepsilon/3$.

We set $j = 2^{j'}$ and show that $(\mathbf{P}_k^{(r+1)}(\omega, n))_{n=1}^\infty$ has property P with this j . For all $j < n < \varphi_m(2^j)$ (i.e. $2^{j'} < n < \varphi_{m+1}(2^{j'})$), we have

$$\begin{aligned} \left\| \mathbf{P}_k^{(r+1)}(\omega, n) - \mathbf{q} \right\| &= \left\| \frac{\mathbf{P}_k^{(r)}(\omega, 1) + \mathbf{P}_k^{(r)}(\omega, 2) + \cdots + \mathbf{P}_k^{(r)}(\omega, n)}{n} - \mathbf{q} \right\| \\ &= \left\| \frac{\mathbf{P}_k^{(r)}(\omega, 1) + \mathbf{P}_k^{(r)}(\omega, 2) + \cdots + \mathbf{P}_k^{(r)}(\omega, j')}{n} \right. \\ &\quad \left. + \frac{\mathbf{P}_k^{(r)}(\omega, j'+1) + \mathbf{P}_k^{(r)}(\omega, 2) + \cdots + \mathbf{P}_k^{(r)}(\omega, n) - (n-j')\mathbf{q}}{n} - \frac{j'\mathbf{q}}{n} \right\| \\ &\leq \left\| \frac{\mathbf{P}_k^{(r)}(\omega, 1) + \mathbf{P}_k^{(r)}(\omega, 2) + \cdots + \mathbf{P}_k^{(r)}(\omega, j')}{n} \right\| \\ &\quad + \frac{\left\| \mathbf{P}_k^{(r)}(\omega, j'+1) - \mathbf{q} \right\| + \cdots + \left\| \mathbf{P}_k^{(r)}(\omega, n) - \mathbf{q} \right\|}{n} - \frac{\|j'\mathbf{q}\|}{n} \\ &\leq \frac{j'}{n} + \frac{\varepsilon}{3} \frac{n-j'}{n} + \frac{j'}{n} \leq \frac{j'}{2^{j'}} + \frac{\varepsilon}{3} + \frac{j'}{2^{j'}} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

\square

Lemma 4.5. *The set E is a subset of $\mathbb{E}_k^{(r)}$.*

Proof. We will show, that for any $x \in E$ we also have $x \in \mathbb{E}_k^{(r)}$. To this end, let $x \in E$ and $\omega \in X_{\mathcal{P},\phi}$ be the symbolic expansion of x . Since $(\mathbf{P}_k^{(1)}(\omega, n))_n$ has property P , by Lemma 4.4 we get that $(\mathbf{P}_k^{(r)}(\omega, n))_n$ has property P .

Thus it suffices to show that \mathbf{p} is an accumulation point of $(P_k^{(r)}(\omega, n))_n$ for any $\mathbf{p} \in S_k$. Therefore we fix $h \in \mathbb{N}$ and find a $\mathbf{q} \in \mathbb{D}$ such that

$$\|\mathbf{p} - \mathbf{q}\| < \frac{1}{h}.$$

Since $(P_k^{(r)}(\omega, n))_n$ has property P for any $m \in \mathbb{N}$ we find $j \in \mathbb{N}$ with $j \geq h$ and such that if $j < n < \varphi_m(2^j)$ then $\|P_k^{(r)}(\omega, n) - \mathbf{q}\| < \frac{1}{h}$. Hence let n_h be any integer with $j < n_h < \varphi_m(2^j)$, then

$$\|P_k^{(r)}(\omega, n_h) - \mathbf{q}\| < \frac{1}{h}.$$

Thus each n_h in the sequence $(n_h)_h$ satisfies

$$\|\mathbf{p} - P_k^{(r)}(\omega, n_h)\| \leq \|\mathbf{p} - \mathbf{q}\| + \|P_k^{(r)}(\omega, n_h) - \mathbf{q}\| < \frac{2}{h}.$$

Since $n_h > h$ we may extract an increasing sub-sequence $(n_{h_u})_u$ such that $P_k^{(r)}(\omega, n_{h_u}) \rightarrow \mathbf{p}$ for $u \rightarrow \infty$. Thus \mathbf{p} is an accumulation point of $P_k^{(r)}(\omega, n)$, which proves the lemma. \square

Proof of Theorem 2.4. Since by Lemma 4.3 E is residual in U_∞ and by Lemma 4.5 E is a subset of $\mathbb{E}_k^{(r)}$ we get that $\mathbb{E}_k^{(r)}$ is residual in U_∞ . Again we note that $M \setminus U_\infty$ is the countable union of nowhere dense sets and therefore $\mathbb{E}_k^{(r)}$ is also residual in M . \square

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(M. G. Madritsch) INSTITUTE ÉLIE CARTAN
UNIVERSITÉ DE LORRAINE
B.P. 70239
54506 VANDOEUVRE-LÈS-NANCY CEDEX
FRANCE
E-mail address: `manfred.madritsch@univ-lorraine.fr`