# ASYMPTOTIC NORMALITY OF $b$-ADDITIVE FUNCTIONS ON POLYNOMIAL SEQUENCES IN NUMBER SYSTEMS

MANFRED G. MADRITSCH

Abstract. We consider $b$-additive functions $f$ where $b$ is an algebraic integer over $\mathbb{Z}$. In particular, let $p$ be a polynomial, then we show that the asymptotic distribution of $f(\lfloor p(z)\rfloor)$, where $\lfloor\cdot\rfloor$ denotes the integer part with respect to basis $b$, when $z$ runs through the elements of the ring $\mathbb{Z}[b]$ is the normal law. This is a generalization of results of Bassily and Kátai (for the integer case) and of Gittenberger and Thuswaldner (for the Gaussian integers).

## 1. Introduction

The objective of this paper is the consideration of additive functions in number systems. We start with a simple example of a $q$-additive function: Let $s_q$ denote the sum-of-digits function in base $q$, where $q$ is a positive integer. This function has been studied by several authors and we want to mention Delange [5]. He computed the average of the sum-of-digits function, *i.e.*,

$$\frac{1}{N}\sum_{n\leq N} s_q(n)=\frac{q-1}{2}\log_q(N)+\gamma_1(\log_q(N)),$$

where $\log_q$ denotes the logarithm in base $q$ and $\gamma_1$ is a continuous function of period 1.

A canonical question is the distribution into residue classes of the sum-of-digits functions, *i.e.*, considerations of the majority of sets of the form

$$S_{r,m}(N)=\{n\leq N : s_q(n)\equiv r \bmod m\}.$$

In this field Mauduit and Sárközy [25] were able to show the following.

**Theorem.** *Let $\mathcal{A},\mathcal{B}\subset\{1,\dots,N\}$ with $N\in\mathbb{N}$. Then the estimate*

$$\left|\#\{(a,b)\in\mathcal{A}\times\mathcal{B} : a+b\in S_{r,m}(2N)\}-\frac{|\mathcal{A}|\,|\mathcal{B}|}{m}\right|=\mathcal{O}(N^\theta(|\mathcal{A}|\,|\mathcal{B}|)^{\frac12})$$

*holds, where $\theta<1$ and the implied $\mathcal{O}$-constant is absolute.*

An extension of the results above to general $q$-additive functions was given by Bassily and Kátai [3]. Recall that a function $f$ is said to be $q$-additive if it acts only on the $q$-adic digits, *i.e.*, $f(0)=0$ and

$$f(n)=\sum_{k\geq0} f(a_k(n)q^k)\quad\text{for}\quad n=\sum_{k\geq0}a_k(n)q^k,$$

where $a_k(n)\in\mathcal{N}:=\{0,\dots,q-1\}$ are the *digits* of the $q$-adic expansion. Obviously, $s_q$ is a special $q$-additive function.

The above mentioned distributional result by Bassily and Kátai [3] reads as follows.

**Theorem.** *Let $f$ be a $q$-additive function such that $f(aq^k)=\mathcal{O}(1)$ as $k\to\infty$ and $a\in\mathcal{N}$. Furthermore let*

$$m_{k,q}:=\frac1q\sum_{a\in\mathcal{N}} f(aq^k),\quad \sigma_{k,q}^2:=\frac1q\sum_{a\in\mathcal{N}} f^2(aq^k)-m_{k,q}^2,$$

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*and*

$$M_q(x) := \sum_{k=0}^{N} m_{k,q}, \quad D_q^2(x) = \sum_{k=0}^{N} \sigma_{k,q}^2$$

*with* $N = [\log_q x]$. *Assume that* $D_q(x)/(\log x)^{1/3} \to \infty$ *as* $x \to \infty$ *and let* $p(x)$ *be a polynomial with integer coefficients, degree* $d$ *and positive leading term. Then, as* $x \to \infty$,

$$\frac{1}{x}\#\left\{n < x \middle| \frac{f(p(n)) - M_q(x^d)}{D_q(x^d)} < y\right\} \to \Phi(y),$$

*where* $\Phi$ *is the normal distribution function.*

The aim of this paper is to further generalize these results. Nevertheless we want to mention also results concerning number systems related to substitution automaton, which were considered by Dumont and Thomas [9]. For distribution results of additive functions defined over number systems based on linear recurrences we refer the reader to Drmota and Gajdosik [6]. Relations between number systems such as canonical number systems and shift radix systems are considered by Akiyama *et al.* in [1].

In this paper we focus on generalizations of number systems to rings in algebraic number fields. Let $K$ be an algebraic number field of degree $n$ and denote by $\mathcal{O}_K$ its ring of integers. Furthermore let N and Tr denote the *norm* and *trace* of an element of $K$ over $\mathbb{Q}$, respectively.

Before we start with our considerations we need a definition of number systems in integral domains (*cf.* [21]).

**Definition 1.1.** Let $\mathcal{R}$ be an integral domain, $b \in \mathcal{R}$, and $\mathcal{N} = \{n_1, \dots, n_m\} \subset \mathbb{Z}$. Then we call the pair $(b, \mathcal{N})$ a *number system* in $\mathcal{R}$ if every $g \in \mathcal{R}$ admits a unique and finite representation of the form

$$g = a_0 + a_1 b + \cdots + a_h b^h \quad \text{with} \quad a_i \in \mathcal{N} \quad \text{for} \quad i = 0, \dots, h \tag{1.1}$$

and $a_h \neq 0$ if $h \neq 0$. We call $b$ the *base* and $\mathcal{N}$ the *set of digits*.

If $\mathcal{N} = \mathcal{N}_0 = \{0, 1, \dots, m\}$ for $m \geq 1$ then we call the pair $(b, \mathcal{N})$ a *canonical number system.*

It has been shown by Kovács [20] that $b$ is a base of a canonical number system in $\mathcal{O}_K$ if $\{1, b, \dots, b^{n-1}\}$ is an integral power base. Possible bases for different algebraic number fields have been considered in a series of papers [2, 4, 16, 17, 18].

When extending the number system to the complex plane one has to face effects such as amenability, *i.e.*, there exist two different expansions of one number. In fact, one can construct a graph which characterizes all the amenable expansions. This has been done by Müller *et al.* [27] (with a direct approach) and by Scheicher and Thuswaldner [30] (consideration of the odometer).

Another view on number systems are normal numbers. These are numbers in which expansion every possible block occurs asymptotically equally often. Constructions of such numbers have been considered by Dumont *et al.* [8] and the author in [24, 23]

In this paper we mainly concentrate on additive functions. These are functions that act only on the digits of an expansion. Thus we define additive functions in these number systems as follows.

**Definition 1.2.** Let $(b, \mathcal{N})$ be a number system in a integral domain $R$. A function $f$ is called $b$-*additive* if $f(0) = 0$ and for $g \in R$

$$f(g) = \sum_{k \geq 0} f(a_k(g) b^k), \quad \text{for } g = \sum_{k \geq 0} a_k(g) b^k \quad (a_k(g) \in \mathcal{N}). \tag{1.2}$$

As indicated above the simplest version of an additive function is the sum-of-digits function $s_b$ defined by

$$s_b(g) := \sum_{k \geq 0} a_k(g).$$

A first step towards generalization is the consideration of $b$-additive function in the field of Gaussian rationals. First the result by Delange was extended to that field by Grabner *et al.* in [13].

**Theorem.** *Let $(b,\mathcal{N})$ be a canonical number system in $\mathbb{Z}[i]$. Then we have*

$$\sum_{\substack{\mathrm{N}(z)\leq N\\ z\in\mathbb{Z}[i]}} s_b(z)=\frac{\mathrm{N}(b)-1}{2}\pi N\log_{\mathrm{N}(b)}N+N\gamma_2\left(\log_{\mathrm{N}(b)}N\right)+\mathcal{O}\left(\sqrt{N}\log_{\mathrm{N}(b)}N\right)$$

*where $\gamma_2$ is a continuous function of period* 1.

Also the result by Bassily and Kátai was generalized to number systems in the Gaussian rationals. Gittenberger and Thuswaldner [12] gained the following distribution result.

**Theorem.** *Let $(b,\mathcal{N})$ be a canonical number system in $\mathbb{Z}[i]$. Let $f$ be a $b$-additive function such that $f(cb^k)=\mathcal{O}(1)$ as $k\to\infty$ and $c\in\mathcal{N}$. Furthermore let*

$$m_{k,b}:=\frac{1}{\mathrm{N}(b)}\sum_{c\in\mathcal{N}}f(cb^k),\quad \sigma_{k,b}^2:=\frac{1}{\mathrm{N}(b)}\sum_{c\in\mathcal{N}}f^2(cb^k)-m_{k,b}^2,$$

*and*

$$M_b(x):=\sum_{k=0}^{L}m_{k,b},\quad D_b^2(x)=\sum_{k=0}^{L}\sigma_{k,b}^2$$

*with $L=[\log_{\mathrm{N}(b)}x]$. Assume that $D_b(x)/(\log x)^{1/3}\to\infty$ as $x\to\infty$ and let $p(x)$ be a polynomial of degree $d$ with coefficients in $\mathbb{Z}[i]$. Then, as $N\to\infty$,*

$$\frac{1}{\#\left\{z\in\mathbb{Z}[i]|\mathrm{N}(z)<N\right\}}\#\left\{\mathrm{N}(z)<N\left|\frac{f(p(z))-M_b(N^d)}{D_b(N^d)}<y\right.\right\}\to\Phi(y),$$

*where $\Phi$ is the normal distribution function and $z$ runs over the Gaussian integers.*

This build the base for further considerations of $b$-additive functions in canonical number systems in general. First the result of Delange was considered in arbitrary number fields by Thuswaldner [32]. Furthermore the moments of the sum-of-digits function in algebraic number fields were considered by Gittenberger and Thuswaldner [11].

**Theorem.** *Let $K$ be a number field of degree $n$ and $\mathcal{O}_K$ its ring of integers. Furthermore, let $b$ be a base of a canonical number system. Then*

$$\begin{aligned}\sum_{z\in M(N)}(s_b(z))^d=c_b\left(\frac{|\mathrm{N}(b)|-1}{2}\right)^d N\log_{|\mathrm{N}(b)|}^d N+N\sum_{j=0}^{d-1}\log_{|\mathrm{N}(b)|}^j N\gamma_j(\log_{|\mathrm{N}(b)|}N)\\ +\mathcal{O}\left(N^{\frac{n-1}{n}}\log_{|\mathrm{N}(b)|}^d N\right),\end{aligned}$$

*where $M(N)$ is the set defined below in (2.4), $c_b$ is a constant depending on $K$ and $b$, and the $\gamma_j$s are continuous functions of period* 1.

In the same vain the above mentioned result by Mauduit and Sárközy was generalized to arbitrary number fields by Thuswaldner [33]. Therefore we write

$$U_{r,m}(M(T))=\{z\in M(T): s_b(z)\equiv r\bmod m\},$$

where $M(T)$ is the set described below in (2.4). Then his result reads as follows.

**Theorem.** *Let $K$ be a number field of degree $n$ with ring of integers $\mathcal{O}_K$. Let $b$ be the base of a canonical number system in $\mathcal{O}_K$ and $m_b(x)=x^n+\cdots+b_1x+b_0$ the minimal polynomial of $b$. If $(m_b(1),m)=1$, then*

$$\left|\#\left\{(a,b)\in\mathcal{A}\times\mathcal{B}:a+b\in U_{r,m}(M(2T))\right\}-\frac{|\mathcal{A}|\,|\mathcal{B}|}{m}\right|=\mathcal{O}(|M(T)|^{\theta}\,(|\mathcal{A}|\,|\mathcal{B}|)^{\frac{1}{2}})$$

*holds for any two sets $\mathcal{A},\mathcal{B}\subset M(T)$. Furthermore $\theta<1$ and the implied $\mathcal{O}$-constant is absolute.*

Despite of these considerations of the sum-of-digits function and other $b$-additive functions, we also want to mention Kátai and Liardet [19], who could show a Delange type result for $b$-multiplicative functions. Finally there has also been work on the generalization of Waring's Problem restricted to sets of the form $U_{r,m}$ defined above. Here we want to mention Pethő and Tichy [28] (counting the number of solutions for a $S$-unit equation) and Thuswaldner and Tichy [34] (counting the number of solutions of Waring's Problem with digital restrictions).

Since for a ring of integers to have a power integral basis is a quite strong assumption we want to consider more general settings in this paper. It was shown by Kovács and Pethő [22] that there are number systems in rings of the form $\mathbb{Z}[\beta]$ with $\beta$ an algebraic integer. The main problem we have to face is the different setting for these number systems. First of all this ring need not to be the ring of integers, however, this we can circumvent by considering the ring in relation to the integral closure of $\mathbb{Q}(\beta)$. Secondly the Weyl sums in algebraic number fields are motivated by consideration of Waring's Problem. In our case, however, the length of expansion depends on the absolute value of the conjugates of the base. These may not be equal and therefore we have to slightly modify the Weyl sums in order to meet our conditions. This will be established in Section 3 where we develop a more general estimation of these sums.

## 2. Definitions and Results

In the following paragraphs we will define the tools we need in order to properly estimate the distribution. These definitions deal with algebraic number fields and their relatives and are standard in the area and the reader may refer to Ribenboim [29] or Wang [35].

Throughout the rest of the paper we fix an algebraic integer $\beta$ of degree $n$ over $\mathbb{Z}$. Then we set $K=\mathbb{Q}(\beta)$ to be an algebraic number field and denote by $\mathcal{O}_K$ its ring of integers (aka its maximal order). Furthermore we set $R=\mathbb{Z}[\beta]$ to be our ring of consideration. Then let $K^{(\ell)}$ $(1\leq \ell\leq r_1)$ be the real conjugates of $K$, while $K^{(m)}$ and $K^{(m+r_2)}$ $(r_1<m\leq r_1+r_2)$ are the complex conjugates of $K$, where $r_1+2r_2=n$. Throughout this paper the indices $\ell$ and $m$ are always over the sets of integers cited here. Furthermore we set $r=r_1+r_2$ and call the pair $(r_1,r_2)$ the signature of $K$.

If not stated otherwise an upper case letter will always denote a real number, a lower case letter an element of $\mathbb{Z}[\beta]$ or $\mathcal{O}_K$ and a Greek letter an element of $\overline{K}$, the completion of $K$. Furthermore sums are always extended over rational or algebraic integers, respectively.

For $\gamma\in K$ we denote by $\gamma^{(i)}$ $(1\leq i\leq n)$ the conjugates of $\gamma$. In order to extend the term of conjugation to the completion $\overline{K}$ of $K$ we define for $\gamma_j\in K$ and $x_j\in\mathbb{R}$ $(1\leq j\leq n)$ $\lambda=\sum_{1\leq j\leq n}x_j\gamma_j$ and $\lambda^{(i)}:=\sum_{1\leq j\leq n}x_j\gamma_j^{(i)}$. We recall that for $\lambda\in\overline{K}$

$$\mathrm{N}(\lambda)=\prod_{1\leq i\leq n}\lambda^{(i)},\quad \mathrm{Tr}(\lambda)=\sum_{1\leq i\leq n}\lambda^{(i)},\quad \overline{|\lambda|}=\max_{1\leq i\leq n}\left|\lambda^{(i)}\right| \tag{2.1}$$

are the *norm*, *trace* and *house* of an element of $\overline{K}$ over $\mathbb{Q}$, respectively. Furthermore for $\lambda\in\overline{K}$ let

$$e(x):=\exp(2\pi i\,x)\quad\text{and}\quad E(\lambda)=e(\mathrm{Tr}(\lambda)). \tag{2.2}$$

Let $\delta$ be the *different*, $\Delta=\Delta_{K|\mathbb{Q}}$ the absolute value of the *discriminant* of $K$ over $\mathbb{Q}$, and $D$ be the absolute value of the discriminant of $R$ over $\mathbb{Z}$ (as $\mathbb{Z}$-module).

We will need some geometry of numbers and therefore let $\eta_1,\dots,\eta_n$ be a basis of $R$ as $\mathbb{Z}$-module and $\omega_1,\dots,\omega_n$ be an integral basis of $\mathcal{O}_K$. Furthermore we let $\rho_1,\dots,\rho_n$ be the corresponding basis of $\delta^{-1}$ such that

$$\mathrm{Tr}\left(\rho_i\omega_j\right)=\begin{cases}1 & i=j,\\ 0 & i\neq j.\end{cases}$$

Let $\Lambda$ be the lattice constructed by the $n$ linear forms

$$L_i=\sum_{j=1}^{n}\rho_j^{(i)}x_j.$$

Then we denote by $\lambda_1$ the first successive minimum of the convex body $B:=\{z\in\mathbb{R}^n:|z|\leq 1\}$ with respect to the lattice $\Lambda$.

We call a number totally non-negative if $\lambda^{(i)} \geq 0$ for $1 \leq i \leq n$. As we will successively extend the maximum length of the expansions we define $N(T_1, \dots, T_r)$ to be the set

$$N(T_1, \dots, T_r) := \left\{ \lambda \in R : \left| \lambda^{(i)} \right| \leq T_i, 1 \leq i \leq r \right\}. \tag{2.3}$$

In the same manner we will need the corresponding set of integers in $\mathcal{O}_K$. Thus

$$M(T_1, \dots, T_r) := \left\{ \lambda \in \mathcal{O}_K : \left| \lambda^{(i)} \right| \leq T_i, 1 \leq i \leq r \right\}. \tag{2.4}$$

We give a characterization of number systems in $R$.

**Lemma 2.1** ([21, Theorem 3])**.** *Let $b \in \mathbb{Z}[\beta]$, $\mathcal{N} \subset \mathbb{Z}$, and $A = \max_{a \in \mathcal{N}} |a|$. Then $(b, \mathcal{N})$ is a number system in $\mathbb{Z}[\beta]$ if and only if*

1. $\left| b^{(i)} \right| > 1$ *for* $i = 1, \dots, n$ *and* $b^{(i)} < -1$ *for* $1, \dots, r_1$,
2. $\mathcal{N}$ *is a full residue system modulo* $|\mathrm{N}(b)|$ *with* $0 \in \mathcal{N}$,
3. $\beta \in \mathbb{Z}[b]$,
4. *all* $\eta \in \mathbb{Z}[\beta]$ *with*
$$\left| \eta^{(i)} \right| \leq \frac{A}{\left| b^{(i)} \right| - 1} \quad (i = 1, \dots, n),$$
*have a representation in* $(b, \mathcal{N})$.

Since we want to run over the integers in $R$ with respect to an increasing length of expansion (1.1), we have to consider the relation of length and the absolute value of an element.

**Lemma 2.2** ([22, Theorem])**.** *Let $\ell(\gamma)$ be the length of the expansion of $\gamma$ to the base $b$. Then*

$$\left| \ell(\gamma) - \max_{1 \leq i \leq n} \frac{\log \left| \gamma^{(i)} \right|}{\log \left| b^{(i)} \right|} \right| \leq C.$$

Thus we fix a $T$ and set $T_i$ for $1 \leq i \leq n$ such that

$$\log T_i = \log T \frac{\log \left| b^{(i)} \right|^n}{\log |\mathrm{N}(b)|}. \tag{2.5}$$

In view of Lemma 2.2 we get that the expansions of the elements of $N(\mathbf{T})$ have the same maximum length. Furthermore we will write for short $M(\mathbf{T}) := M(T_1, \dots, T_r)$ and $N(\mathbf{T}) := N(T_1, \dots, T_r)$ with $T_i$ as in (2.5).

Finally one can extend the definition of a number system also for negative powers of $b$. Then for $\gamma \in \overline{K}$ such that

$$\gamma = \sum_{i=-\infty}^{h} a_i b^i \quad \text{with} \quad a_i \in \mathcal{N}$$

we call

$$\lfloor \gamma \rfloor := \sum_{i=0}^{h} a_i b^i \quad \text{and} \quad \{\gamma\} := \sum_{i \geq 1} a_i b^{-i}$$

the *integer part* and *fractional part* of $\gamma$,respectively.

Now we are in the position to state our main result.

**Theorem 2.3.** *Let $(b, \mathcal{N})$ be a number system in $R$ and $f$ be a $b$-additive function such that $f(ab^k) = \mathcal{O}(1)$ as $k \to \infty$ and $a \in \mathcal{N}$. Furthermore let*

$$m_{k,b} := \frac{1}{\mathrm{N}(b)} \sum_{a \in \mathcal{N}} f(ab^k), \quad \sigma_{k,b}^2 := \frac{1}{\mathrm{N}(b)} \sum_{a \in \mathcal{N}} f^2(ab^k) - m_{k,b}^2,$$

*and*

$$M_b(x) := \sum_{k=0}^{L} m_{k,q}, \quad D_b^2(x) = \sum_{k=0}^{L} \sigma_{k,q}^2$$

*with* $L=[\log_{\mathrm{N}(b)} x]$. *Assume that there exists an* $\varepsilon>0$ *such that* $D_b(x)/(\log x)^\varepsilon\to\infty$ *as* $x\to\infty$ *and let* $p\in\overline{K}[X]$ *be a polynomial of degree* $d$. *Then, as* $T\to\infty$ *let* $T_i$ *be as in* (2.5),

$$\frac{1}{\#N(\mathbf{T})}\#\left\{z\in N(\mathbf{T})\left|\frac{f(\lfloor p(z)\rfloor)-M_b(T^d)}{D_b(T^d)}<y\right.\right\}\to\Phi(y),$$

*where* $\Phi$ *is the normal distribution function.*

We will show this theorem essentially in five steps.

(1) We start with the estimation of the Weyl sums which will occur in the proof. First estimates are provided by generalizations of Waring's Problem to algebraic number fields together with Hua's method of estimating Weyl sums. We will tune these tools in order to meet our requirements in Section 3. The main difference will be the approximation of the coefficients.
(2) For counting the number of occurrences of a certain digit in the expansion of the integer values of the polynomial $p$ we need an Urysohn function. This function is developed in Section 4 where we consider the fundamental domain of the number system.
(3) The error term when counting the number of occurrences of a digit comes from those which are near the border of the fundamental domain. Therefore we estimate the number of elements which are in a small tube around the border in Section 5.
(4) In Section 6 we have to count the number of elements within the fundamental domain itself, which gives the proof of the central proposition.
(5) Finally we show Theorem 2.3 by truncation and two applications of the Fréchet-Shohat Theorem.

## 3. Weyl sums

In this section we want to consider and estimate the exponential sums which will occur in the following sections. We will begin by giving some background on how exponential sums are defined in number fields.

Let a boldface letter always denote a vector. Then for $\mathbf{T}=(T_1,\dots,T_r)$ we set $N(\mathbf{T})=N(T_1,\dots,T_r)$. We call the sum

$$\sum_{x\in N(\mathbf{T})}E(g(x)),$$

where $E$ is as in (2.2) and $g$ is a polynomial, a *Weyl sum*. In the generalizations of Waring's Problem the set $N(\mathbf{T})$ is replaced by the set $M(\mathbf{T})$, but one can in fact replace the set by any finite set.

The main tool in order to estimate these sums is Weyl's differentiation method. Therefore we need a generalization of Dirichlet's theorem on rational approximation, which is provided by Siegel (*cf.* [31]). Since in our case the $T_i$ are not all equal (see Lemma 2.2) we have to slightly modify Siegel's original theorem in order to cope with this new situation.

**Lemma 3.1.** *Let* $N_1,\dots,N_r$ *be real numbers and let* $N=\sqrt[n]{N_1\cdots N_{r_1}(N_{r_1+1}\cdots N_{r_1+r_2})^2}$ *be their geometric mean. Suppose that* $N>\Delta^{\frac1n}$, *then, corresponding to any* $\xi\in K$, *there exist* $q\in\mathcal{O}_K$ *and* $a\in\delta^{-1}$ *such that*

$$\left|q^{(i)}\xi^{(i)}-a^{(i)}\right|<N_i^{-1},\quad 0<\left|q^{(i)}\right|\le N_i,\quad 1\le i\le r,$$
$$\max\left(N_i\left|q^{(i)}\xi^{(i)}-a^{(i)}\right|,\left|q^{(i)}\right|\right)\ge\Delta^{-\frac12},\quad 1\le i\le r,$$

*and*

$$\mathrm{N}((q,a\delta))\le\Delta^{\frac12}.$$

*Proof.* This easily follows from the appropriate modifications in the proof of Theorem 3.1 of [35]. □

Now we can state the main estimate of the exponential sum above.

**Proposition 3.2.** *Let*

$$T=\sqrt[n]{T_1\cdots T_{r_1}(T_{r_1+1}\cdots T_{r_1+r_2})^2}$$

*be the geometric mean of the* $T_i$ $(1\leq i\leq r)$. *Suppose that*

$$g(x)=\alpha_d x^d+\cdots+\alpha_1 x \tag{3.1}$$

*is a polynomial of degree* $d$. *If for the leading coefficient* $\alpha_d$ *there exist* $a\in\delta^{-1}$ *and* $q\in\mathcal{O}_K$ *as in Lemma 3.1 with* $N_i=T_i^d(\log T)^{-\sigma_1}$ *and*

$$(\log T)^{\sigma_1}\leq\left|q^{(i)}\right|\leq T_i^d(\log T)^{-\sigma_1}\quad 1\leq i\leq r,$$

*then*

$$\sum_{x\in N(\mathbf{T})}E(g(x))\ll T^n(\log T)^{-\sigma_0}$$

*with* $\sigma_1\geq 2^{d-1}\left(\sigma_0+r2^{2d}\right)$.

Since in our considerations the $T_i$ need not all be equal, we need the $(\log T)^{-\sigma_0}$ term, and the sum is extended over a different set, we have to modify the proof of Theorem 3.2 of Wang [35]. We start with the main tools needed for the proof of Proposition 3.2.

The first tool deals with the different set over which the sum is extended. It also provides a relation between the number of elements of the sets $M(\mathbf{T})$ and $N(\mathbf{T})$.

**Lemma 3.3.** *Let* $T_i$ $(1\leq i\leq r)$ *be positive integers and set* $T_{r_1+r_2+i}=T_{r_1+i}$ *for* $(1\leq i\leq r_2)$. *Then*

$$\#M(\mathbf{T})=\frac{2^r\pi^{r_2}}{\sqrt{\Delta}}T_1\cdots T_n+\mathcal{O}\left(T_0^{n-1}\right),$$
$$\#N(\mathbf{T})=\frac{2^r\pi^{r_2}}{\sqrt{D}}T_1\cdots T_n+\mathcal{O}\left(T_0^{n-1}\right),$$

*where* $T_0=\max\left(1,(T_1\cdots T_n)^{1/n}\right)$.

*Proof.* The estimation of $\#M(\mathbf{T})$ is Lemma 3.2 of [35] and the second estimate follows easily by modifications of the lattice in this proof. □

Since in the classical case the $T_i$ are all equal we have to rewrite the corresponding tools in the proof of Wang's Lemma 3.6. Therefore we need the following adoption of Lemma 3.5 of Wang [35].

**Lemma 3.4.** *Let* $T_i$ *and* $N_i\geq 0$ *for* $1\leq i\leq s$ *be integers. Then denote by* $M$ *the set of all points* $(t_1,\ldots,t_s)\in\mathbb{Z}^s$ *such that*

$$T_i\leq t_i<T_i+N_i\quad 1\leq i\leq s.$$

*Let* $M_0$ *be a subset of* $M$ *and define*

$$S=\sum_{\mathbf{t}\in M_0}\min\left(\frac{N_1}{t_1},\cdots,\frac{N_s}{t_s}\right).$$

*Then*

$$S\ll N(\#M_0)^{1-\frac{1}{s}}(\log(N+2)),$$

*where* $N$ *is the geometric mean of the* $N_i$.

*Proof.* This proof mainly follows that of Lemma 3.5 of Wang [35]. In the same way we start by setting $M_\nu$ to be the subset of $M_0$ such that $\frac{t_\nu}{N_\nu}\geq\frac{t_i}{N_i}$ for $1\leq i\leq s$. Furthermore we denote by $A_\nu=\#M_\nu$ its number of elements and by

$$S_\nu=\sum_{\mathbf{t}\in M_\nu}\min\left(\frac{N_1}{t_1},\cdots,\frac{N_s}{t_s}\right)$$

the restriction of the sum $S$ to elements of $M_\nu$.

Then it suffices to show that

$$S_\nu\ll NA_\nu^{1-\frac{1}{s}}\log(N_\nu+2),$$

which together with $S\leq S_1+\cdots+S_s$ proves the lemma.

Without loss of generality we show this for $\nu=1$. For $t>0$ let $D(t)$ be the subset of $\mathbf{u}\in\mathbb{R}^s$ such that
$$t\geq u_1>0,\quad \frac{u_1}{N_1}\geq\frac{u_i}{N_i}\geq 0,\quad 2\leq i\leq s.$$
Let $M(t):=D(t)\cap\mathbb{Z}^s$ be the integer points in $D(t)$ and denote by $n(t)$ their number. In the same manner let $M_0(t):=\{\mathbf{t}\in M_0 : t\geq t_1\}$ and denote by $n_0(t)=\#M_0(t)$ its cardinality.

Now let $t_0$ be an integer such that
$$n(t_0)\leq A_1=n_0(N_1)<n(t_0+1).$$
Then
$$S_1\leq\sum_{\mathbf{t}\in M(t_0+1)}\frac{N_1}{t_1}\leq N_1\prod_{i=2}^{s}\frac{N_i}{N_1}(t_0+2)\sum_{t_1\leq t_0+1}\frac{1}{t_1}\leq\frac{N^s(t_0+2)^{s-1}}{N_1^{s-1}}\log(t_0+2)$$
and
$$A_1\geq n(t_0)\geq\sum_{t=1}^{t_0}\prod_{i=2}^{s}\frac{N_i}{N_1}t=\frac{N^s}{N_1^s}\sum_{t=1}^{t_0}t^{s-1}\geq c(s)\frac{N^s}{N_1^s}t_0^s.$$
Putting these together proves the lemma. □

Now we state our modified version of Lemma 3.6 of Wang [35]. In its original version this lemma essentially goes back to Mitsui [26].

**Lemma 3.5.** *Let $A$,$B_i$ $(1\leq i\leq r)$,$N$ be positive numbers satisfying $A\geq 1$ and $N\gg 1$. Furthermore let $B$ be the geometric mean of the $B_i$ and let $N_i$ $(1\leq i\leq r)$ be in the same ration to $N$ as the $B_i$ are to $B$, i.e.,*
$$B:=\sqrt[n]{B_1\cdots B_{r_1}(B_{r_1+1}\cdots B_r)^2}\quad\text{and}\quad N_i:=N\frac{B_i}{B}\quad 1\leq i\leq r.$$
*Suppose that $1\leq B\ll N$, then, for any $\xi\in K$,*
$$\sum_{m\in M(\mathbf{B})}\min_{1\leq i\leq n}\left(A,|1-E(\xi m\eta_i)|^{-1}\right)\ll AB^n\left(\frac{1}{|\mathrm{N}(q)|}+\frac{1}{B}+\frac{N\log N}{AB}+\frac{\log N}{A}\right),$$
*where $q$ denotes an integer of $K$ satisfying the conditions in Lemma 3.1 with $\xi$ and $N_i$.*

*Proof.* Since this is a modification of the proof of Lemma 3.6 in Wang [35] we only sketch the proof and mainly follow the lines there. We also try to use the same naming.

First we have to mention that $\eta_i$ is an element of the basis of $R$ (and not of $\mathcal{O}_K$ as in Wang's proof). But this provides us with no difficulty, since this element is fixed throughout the whole summation.

Let $\mathbf{X}$ be the Minkowski embedding, *i.e.*, for $\xi\in\overline{K}$
$$\mathbf{X}(\xi)=(X_1(\xi),\dots,X_n(\xi)):=(\xi^{(1)},\dots,\xi^{(r_1)},\Re(\xi^{(r_1+1)}),\dots,\Im(\xi^{(r_1+r_2)})).$$
Then for each $m\in M(\mathbf{B})$ we write
$$\mathrm{Tr}(\xi m\eta_i)=e_i+d_i\quad(1\leq i\leq n)\quad\text{and set}\quad\zeta=\sum_{i=1}^{n}d_i\rho_i$$
with $e_i$ being rational integers and $-\frac12\leq d_i\leq\frac12$.

One easily checks that
$$S=\sum_{m\in M(\mathbf{B})}\min_{1\leq i\leq n}\left(A,|1-E(\xi m\eta_i)|^{-1}\right)\ll\sum_{m\in M(\mathbf{B})}\min_{1\leq i\leq n}\left(A,|X_i(\zeta)|^{-1}\right)=S^*.$$

Now we assign to every $m\in M(\mathbf{B})$ its corresponding $\zeta$ and a vector $y(m)$ defined by
$$y(m)=(R_1X_1(\zeta),\dots,R_nX_n(\zeta))\quad\text{with}\quad R_i=2\Delta^{\frac1n}\left|q^{(i)}\right|.$$

We set $c_{11}=nc_{10}\Delta^{1/n}$, where $c_{10}$ is a constant such that $c_{11}>\Delta^{1/2}$, and get that

$$|X_i(\zeta)|\leq \frac{1}{2}c_{11}\Delta^{-1/n}.$$

Then we divide the set $\{1,2,\dots,n\}$ into three parts

$$\begin{aligned} J_1&:=\left\{1\leq i\leq n:\frac{B_i}{N_i}\Delta^{\frac{1}{n}}\geq 2c_{11}\left|q^{(i)}\right|\right\},\\ J_2&:=\left\{1\leq i\leq n:\frac{1}{2}\geq 2c_{11}\left|q^{(i)}\right|>\frac{B_i}{N_i}\Delta^{\frac{1}{n}}\right\},\\ J_3&:=\left\{1\leq i\leq n:2c_{11}\left|q^{(i)}\right|>\frac{1}{2}\right\}.\end{aligned}$$

Furthermore we set

$$\tau_i=2\frac{B_i}{N_i}\Delta^{\frac{1}{n}}\text{ for }i\in J_1,\quad \tau_i=4c_{11}\left|q^{(i)}\right|\text{ for }i\in J_2.$$

For the rest we set $\tau_i$ such that $\prod_{i=1}^n\tau_i=2^{-2n}$. Then we divide the set of possible vectors $y(m)$ by defining for every vector $\mathbf{g}\in\mathbb{Z}^n$

$$B(\mathbf{g}):=\left\{\mathbf{x}:\tau_i\left(g_i-\frac{1}{2}\right)\leq x_i<\tau_i\left(g_i+\frac{1}{2}\right),1\leq i\leq n\right\}.$$

By the same lines as in the proof of Wang we get that if there are two $m$ and $m_1$ such that $m,m_1\in B(\mathbf{g})$ then $m-m_1\in\mathfrak{a}$ for a certain ideal $\mathfrak{a}$ with

$$|\mathrm{N}(q)|\leq\Delta^{\frac{1}{n}}\mathrm{N}(\mathfrak{a}).$$

Finally we denote by $W(\mathbf{g})$ the number of $m\in M(\mathbf{B})$ such that $y(m)\in B(\mathbf{g})$. Thus following the lines of the proof of Wang we get that

$$W(\mathbf{g})\ll 1+W_0=1+\prod_{i\in J_1}N_i\prod_{j\in J_2\cup J_3}\frac{B_j}{\left|q^{(j)}\right|}=1+B^n\prod_{i\in J_1}\frac{N_i}{B_i}\prod_{j\in J_2\cup J_3}\left|q^{(j)}\right|^{-1}.$$

Now we split the sum $S^*$ up into two parts where $S_1$ consists of all elements $m\in M(\mathbf{B})$ and $y(m)\in B(\mathbf{0})$ and $S_2$ is the rest.

We start with the estimation of $S_1$ and distinguish two cases according to whether $J_1\cup J_2=\emptyset$ or not.

- For $J_1\cup J_2=\emptyset$ we get as in the proof of Wang that

$$S_1\ll A+\frac{AB^n}{|\mathrm{N}(q)|}.\tag{3.2}$$

- For $J_1\cup J_2\neq\emptyset$ we rewrite the sum and get

$$S_1\ll\sum_{\substack{m\in\mathfrak{a}\\ m\in M(2\mathbf{B})}}\min_{j\in J_1\cup J_2}\left(A,\frac{N_j}{|X_j(m+\xi_0)|}\right).$$

We again divide the area of possible $\mathbf{X}$. For $\mathbf{t}\in\mathbb{Z}^n$ we define

$$B^*(\mathbf{t})=\left\{\mathbf{x}:\frac{\mathrm{N}(\mathfrak{a})}{3}\frac{B_i}{B}\left(t_i-\frac{1}{2}\right)\leq x_i<\frac{\mathrm{N}(\mathfrak{a})}{3}\frac{B_i}{B}\left(t_i+\frac{1}{2}\right)\right\}.$$

We get that $B^*(t)\cup M(2B)$ contains at most one element for every $\mathbf{t}\in\mathbb{Z}^n$. By noting our definition of $N_i$ we rewrite the sum to get

$$S_1\ll\sum_{\mathbf{t}}\min_{j\in J_1\cup J_2}\left(A,\frac{N}{|t_j|\,\mathrm{N}(\mathfrak{a})^{\frac{1}{n}}}\right)\ll AB^n\left(\frac{1}{B}+\frac{N\log N}{AB}\right),$$

where we used Wang's estimations since the sum is the same.

Together with the estimation (3.2) we get for $S_1$ that

$$S_1 \ll AB^n \left( \frac{1}{|\mathrm{N}(q)|} + \frac{1}{B} + \frac{N \log N}{AB} \right). \tag{3.3}$$

Now we are left with estimating $S_2$. Therefore we distinguish the two cases according to whether $W_0 > 1$ or not.

- For the first case ($W_0 > 1$) we get by following the lines of the proof of Wang that

$$S_2 \ll B^n \log N. \tag{3.4}$$

- In the case of $W_0 \leq 1$ we reach the estimate

$$S_2 \ll \sum_{\mathbf{g}_i \in G_0} \min_{i \in J_3} \left( \frac{N_i}{\left|q^{(i)}\right|} \right),$$

where $G_0$ is the set of all $g_i, i \in J_3$ such that

$$W(\mathbf{g}) \neq 0, \quad \{g_i\} \neq \mathbf{0}.$$

In the same manner as in Wang's proof we get that the value $|g_i|$ in $G_0$ does not exceed $N_i$. Thus by an application of Lemma 3.4 we get that

$$S_2 \ll NB^{n-1} \log N.$$

Together with the estimation (3.4) this yields

$$S_2 \ll AB^n \left( \frac{\log N}{A} + \frac{N \log N}{AB} \right). \tag{3.5}$$

Putting the estimates of $S_1$ and $S_2$ in (3.3) and (3.5) together proves the lemma. □

Now we need two tools in order to successively apply Weyl's differentiation method. The first let's us replace the sum over the elements of $N(\mathbf{T})$ by an estimation of a minimum.

**Lemma 3.6** ([35, Lemma 3.8])**.** *Let $m \in \mathcal{O}_K$ and $T$ be the geometric mean of the $T_i$, i.e.,*

$$T = \sqrt[n]{T_1 \cdots T_{r_1} (T_{r_1+1} \cdots T_{r_1+r_2})^2}.$$

*Then*

$$\sum_{h \in N(\mathbf{T})} E(\xi m h) \ll T^{n-1} \min_{1 \leq i \leq n} (T, |1 - E(\xi m \eta_i)|^{-1}),$$

*where $\eta_i$ ($1 \leq i \leq n$) is a basis of $R$.*

*Proof.* The proof follows the one of Lemma 3.8 of [35] together with an application of Lemma 3.3. □

The second one deals with the Weyl's differentiation as such. The main idea is to nest the sum in order to reduce the degree of the involved polynomial by one.

**Lemma 3.7.** *Suppose that $1 \leq t \leq d-1$ and $T$ is the geometric mean of the $T_i$, then we have*

$$\left| \sum_{h \in N(\mathbf{T})} E(g(h)) \right|^{2^t} \ll T^{(2^t - t - 1)n} \sum_{h_1, \ldots, h_t \in M(2T)} \left| \sum_{h \in N(\mathbf{T})} E(h_1 \cdots h_t \overline{g}(h, h_1, \ldots, h_t) \xi) \right|,$$

*where*

$$\overline{g}(h, h_1, \ldots, h_t) = d(d-1) \cdots (d-t+1) \alpha_d h^{d-t} + \cdots$$

*is a polynomial of degree $d - t$ in $h, h_1, \ldots, h_t$.*

*Proof.* By Hölder's inequality we get that

$$\begin{aligned}\left|\sum_{h\in N(\mathbf{T})} E(g(h))\right|^{2^t} &\leq \left(\sum_{h_1\in N(2\mathbf{T})}\left|\sum_{h\in N(\mathbf{T})} E(h_1 g(h_1,h))\right|\right)^{2^{t-1}}\\ &\leq \left(\sum_{h_1\in M(2T)}\left|\sum_{h\in N(\mathbf{T})} E(h_1 g(h_1,h))\right|\right)^{2^{t-1}}\\ &\ll T^{\left(2^{t-1}-1\right)n}\sum_{h_1\in M(2T)}\left|\sum_{h\in N(\mathbf{T})} E(h_1 g(h_1,h))\right|^{2^{t-1}}.\end{aligned}$$

Now we iterate this process in the same manner as in the proof of Lemma 3.9 in [35] to finish the proof. □

Now we consider the divisor function in more detail. This idea essentially goes back to Hua [14].

**Lemma 3.8** ([35, Lemma 3.7])**.** *For $a\in\mathcal{O}_k$ and $T\geq 0$ let $d_k(a,T)$ be the number of solutions of the equation*

$$u_1\cdots u_k = a,\quad \text{where}\quad u_1,\ldots,u_k\in M(T)\ \text{for}\ i=1,\ldots,k.$$

*Then for every $\varepsilon>0$*

$$d_k(a,T)\ll |\mathrm{N}(a)|^{\varepsilon}\,(\log T)^{(r-1)(k-1)}.$$

We write for short $d(a,T) := d_2(a,T)$. Now we have collected all our tools in order to estimate sums involving divisor function, which will occur in our proof of Proposition 3.2.

**Lemma 3.9.** *Let $t$ be a non-negative integer and $\mathbf{T}=(T_1,\ldots,T_r)$. Then*

$$\sum_{m\in M(\mathbf{T})}\frac{(d(m,T_0))^t}{\mathrm{N}(m)}\ll (n(\log T_0)^r)^{2^t},$$

*where $T_0=\max\left(1,\left(T_1\cdots T_{r_1}T_{r_1+1}^2\cdots T_r^2\right)^{1/n}\right)$.*

*Proof.* For simplicity we set $T_{r_1+r_2+i}=T_{r_1+i}$ for $1\leq i\leq r_2$ and continue by induction on $t$. For $t=0$ we get that $m\in M(\mathbf{T})$ implies that $|\mathrm{N}(m)|\leq\prod_{i=1}^n T_i\ll T_0^n$. Thus by Lemma 3.8

$$\sum_{m\in M(\mathbf{T})}\frac{1}{|\mathrm{N}(m)|}\ll\sum_{N\leq T_0^n}\frac{(\log T_0)^{r-1}}{N}\ll n(\log T_0)^r.$$

Now we assume that the lemma holds for $t-1$. Then

$$\begin{aligned}\sum_{m\in M(\mathbf{T})}\frac{(d(m,T_0))^t}{|\mathrm{N}(m)|} &= \sum_{m\in M(\mathbf{T})}\frac{(d(m,T_0))^{t-1}}{|\mathrm{N}(m)|}\sum_{\substack{uv=m\\ u,v\in M(T_0)}}1 \leq \sum_{u\in M(T_0)}\sum_{\substack{uv=m\\ m\in M(\mathbf{T})}}\frac{(d(m,T_0))^{t-1}}{|\mathrm{N}(m)|}\\ &= \sum_{u\in M(\mathbf{T})}\ \sum_{v\in M\left(T_1\left|u^{(1)}\right|^{-1},\ldots,T_r\left|u^{(r)}\right|^{-1}\right)}\frac{(d(uv,T_0))^{t-1}}{|\mathrm{N}(u\cdot v)|}\\ &\ll (n\cdot(\log T_0)^r)^{2^{t-1}}(n\cdot(\log T_0)^r)^{2^{t-1}} = (n\cdot(\log T_0)^r)^{2^t}.\end{aligned}$$

□

By the last lemma we can estimate a divisor sum which occurs in the estimation of our Weyl sum.

**Lemma 3.10.** *Let $t$ be a non-negative integer and let $\mathbf{T}=(T_1,\ldots,T_r)$. Furthermore set $T_{r_1+r_2+i}=T_{r_1+i}$ for $1\leq i\leq r_2$ and suppose that $T_i\ll T_0$ for $1\leq i\leq n$. Then*

$$\sum_{m\in M(\mathbf{T})}(d(m,T_0))^t\ll T_1\cdots T_n(n(\log T_0)^r)^{2^t-1}.$$

*Proof.* We show this by induction on $t$. For $t=0$ this essentially is Lemma 3.3. Now we assume that the lemma holds for $t-1$, then by an application of Lemma 3.9

$$\begin{aligned}\sum_{m\in M(\mathbf{T})}(d(m,T_0))^t &= \sum_{m\in M(\mathbf{T})}(d(m,T_0))^{t-1}\sum_{\substack{uv=m\\ u,v\in M(T_0)}}1=\sum_{u\in M(\mathbf{T})}\sum_{\substack{uv=m\\ u\in M(\mathbf{T})}}(d(m,T_0))^{t-1}\\ &=\sum_{u\in M(\mathbf{T})}\sum_{v\in M\left(T_1\left|u^{(1)}\right|^{-1},\dots,T_r\left|u^{(r)}\right|^{-1}\right)} d(uv,T_0))^{t-1}\\ &\ll \sum_{u\in M(\mathbf{T})}(d(u,T_0))^{t-1}\frac{T_1\cdots T_n}{|\mathrm{N}(u)|}(n(\log T_0)^r)^{2^{t-1}-1}\\ &\ll T_1\cdots T_n(n(\log T_0)^r)^{2^t-1}.\end{aligned}$$

□

Finally we need a Lemma whose idea essentially goes back to Hua (*c.f.* Hilfssatz 6.1 of [14]) in order to get a better estimation of the Weyl sum.

**Lemma 3.11.** *Let $t$ be a non-negative integer and let $\mathbf{T}=(T_1,\dots,T_r)$. Suppose that $T_i\ll T_0$ for $1\le i\le n$. Then, for any $\sigma_2\ge 2^{3t-1}$ we get*

$$\sum_{m\in M(\mathbf{T})}{}'(d(m,T_0))^t\ll T_0^n(n(\log T_0)^r)^{-\sigma_2}$$

*where $\sum'$ stands for the sum over all $m$ such that*

$$(n(\log T_0)^r)^{\sigma_2}\ll(d(m,T_0))^t.$$

*Proof.*

$$\begin{aligned}(n(\log T_0)^r)^{2\sigma_2}\sum_{m\in M(\mathbf{T})}{}'(d(m,T_0))^t&\ll\sum_{m\in M(\mathbf{T})}(d(m,T_0))^{3t}\\ &\ll T_0^n(n(\log T_0)^r)^{2^{3t}-1}\ll T_0^n(n(\log T_0)^r)^{\sigma_2}.\end{aligned}$$

□

*Proof of Proposition 3.2.* We set $G=2^{d-1}$ and get by an application of Lemma 3.7

$$\left|\sum_{m\in N(\mathbf{T})}E(g(m))\right|^G\ll T^{(G-d)n}\sum_{h_1,\dots,h_{d-1}\in M(2T)}\left|\sum_{h\in M(\mathbf{T})}E(\alpha_d mh)\right|,$$

where

$$m=d!h_1\cdots h_{d-1}. \tag{3.6}$$

Now we denote by $A(m)$ the number of solutions of (3.6). Noting that $d_k(m,T)\le d(m,T)^k$ we get that

$$A(m)\ll\begin{cases}T^{(d-2)n} & \text{if } m=0,\\ d(m,T)^{d-1} & \text{if } m\neq 0.\end{cases}$$

Putting everything together yields

$$\left|\sum_{m\in N(\mathbf{T})}E(g(m))\right|^G\ll T^{(G-2)n}+T^{(G-d)n}\sum_{m\in M(d!2^qT^{d-1})}d(m,T)^{d-1}\left|\sum_{h\in M(\mathbf{T})}E(\alpha_d mh)\right|.$$

Now we distinguish two cases for $m$ according to the hypotheses of Lemma 3.11, *i.e.*, whether $(n(\log T)^r)^{\sigma_2} \ll d(m,T)$ or not. Thus by an application of Lemma 3.5 and Lemma 3.6

$$\begin{aligned}\left|\sum_{m\in N(\mathbf{T})} E(g(m))\right|^G &\ll T^{(G-2)n} + T^{(G-d)n}\left(T^n(n(\log T)^r)^\sigma + {\sum_m}' (n(\log T)^r)^\sigma \left|\sum_h E(\alpha_d mh)\right|\right) \\ &\ll T^{(G-2)n} + T^{(G-d+1)n}(\log T)^{r\sigma} \\ &\quad + T^{(G-d)n}(\log T)^{r\sigma} {\sum_m}' T^{n-1}\min_i \left(T, |1-E(\alpha_d mh)|^{-1}\right) \\ &\ll T^{(G-2)n} + T^{(G-d+1)n}(\log T)^{r\sigma} + T^{Gn}(\log T)^{r\sigma}\left(\frac{1}{\mathrm{N}(\alpha_d)} + \frac{1}{T} + (\log T)^{-\sigma_1}\right) \\ &\ll T^{(G-2)n} + T^{(G-d+1)n}(\log T)^{r\sigma} + T^{Gn}(\log T)^{r\sigma-\sigma_1}.\end{aligned}$$

□

## 4. Fundamental domain

In this section we want to construct the Urysohn function for indicating the elements starting with a certain digit. The following definitions are standard in that area and we mainly follow Gittenberger and Thuswaldner [12]. We need the fundamental domain, which is defined as the set of all numbers whose integer part is zero, *i.e.*,

$$\mathcal{F}' := \left\{ z \in \mathbb{C} \,\middle|\, z = \sum_{k\geq 1} a_k b^{-k}, a_k \in \mathcal{N} \right\}.$$

It is more convenient to consider the embedding of the fundamental domain in $\mathbb{R}^n$. We note that if $(b,\mathcal{N})$ is a number system then $b$ is also an algebraic integer of degree $n$ and $K = \mathbb{Q}(\beta) = \mathbb{Q}(b)$ (*cf.* [16, 21]). Thus we get that $\{1, b, \ldots, b^{n-1}\}$ is an $\mathbb{Z}$-basis for $\mathbb{Z}[\beta]$, a $\mathbb{Q}$-basis of $K$ and an $\mathbb{R}$-basis of $\overline{K}$. We may define the embedding $\phi$ by

$$\phi : \left\{ \begin{array}{ccc} \overline{K} & \to & \mathbb{R}^n, \\ \alpha_1 + \alpha_2 b + \cdots + \alpha_n b^{n-1} & \mapsto & (\alpha_1, \ldots, \alpha_n). \end{array} \right. ,$$

where $\overline{K}$ is the completion of $K$.

Now let $m_b(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ be the minimal polynomial of $b$, then we define the corresponding matrix $B$ by

$$B = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & -a_0 \\ 1 & 0 & \cdots & \cdots & 0 & \vdots \\ 0 & 1 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}. \tag{4.1}$$

One easily checks that

$$\phi(b \cdot z) = B \cdot \phi(z).$$

By this we define the embedding of the fundamental domain by

$$\mathcal{F} = \phi(\mathcal{F}') = \left\{ z \in \mathbb{R}^n \,\middle|\, z = \sum_{k\geq 1} B^{-k} a_k, a_k \in \phi(\mathcal{N}) \right\}.$$

Furthermore we note that by Theorem 1 of [16]

$$\lambda((\mathcal{F} + g_1) \cap (\mathcal{F} + g_2)) = 0$$

for every $g_1, g_2 \in \mathbb{Z}^n$ with $g_1 \neq g_2$, where $\lambda$ denotes the $n$ dimensional Lebesgue measure. Thus $(B, \phi(\mathcal{N}))$ is a matrix number system and a so called *just touching covering system* and we are allowed to apply the results of the paper by Müller et al. [27].

In the rest we combine the ideas of Gittenberger and Thuswaldner [12] with the results of Kátai and Kőrnyei [16] and Müller et al. [27]. Therefore we only show the results and left the proofs to the reader. For the proper counting of the elements with the same digit in their expansion the border of the fundamental domain is of special interest. In particular, its diameter will provide us with a parameter we need in order to properly estimate the Fourier series of the constructed Urysohn function. We can approximate $\mathcal{F}$ with help of the sets

$$\begin{aligned} Q_0 &:= \left\{ z \in \mathbb{R}^n \middle| \left\| z \right\|_\infty \leq \tfrac{1}{2} \right\}, \\ Q_k &:= \bigcup_{a \in \mathcal{N}} B^{-1}(Q_{k-1} + \phi(a)). \end{aligned}$$

This approximation satisfies $d(\partial Q_k, \partial \mathcal{F}) \ll |b|^{-k}$, where $d(\cdot, \cdot)$ denotes the Hausdorff metric. By consulting [27], we get that the $Q_k$ consists of $|\mathcal{N}|^k$ parallelograms and that there exists a $\mu$ with $1 < \mu < |\mathrm{N}(b)|$ such that $\mathcal{O}(\mu^k)$ of theses parallelograms intersect the boundary of $Q_k$.

Now we define the fundamental domain consisting of all numbers whose first digit equals $a \in \mathcal{N}$, *i.e.*,

$$\mathcal{F}_a = B^{-1}(\mathcal{F} + \phi(a)).$$

Imitating the proof of Lemma 3.1 of [12] we get the following.

**Lemma 4.1.** *For all $a \in \mathcal{N}$ and all $k \in \mathbb{N}$ there exists an axe-parallel tube $P_{k,a}$ with the following properties:*

- *$\partial \mathcal{F}_a \subset P_{k,a}$ for all $k \in \mathbb{N}$,*
- *the Lebesgue measure of $P_{k,a}$ is a $\mathcal{O}\left(\frac{\mu^k}{|\mathrm{N}(b)|^k}\right)$,*
- *$P_{k,a}$ consists of $\mathcal{O}(\mu^k)$ axe-parallel rectangles, each of which has Lebesgue measure $\mathcal{O}(|\mathrm{N}(b)|^k)$,*

*where $\lambda$ denotes the Lebesgue measure.*

As in the proof of Lemma 3.1 of [12] we can construct for each pair $(k, a)$ an axe-parallel polygon $\Pi_{k,a}$ and the corresponding tube

$$P_{k,a} := \left\{ z \in \mathbb{R}^n \middle| \left\| z - \Pi_{k,a} \right\|_\infty \leq 2c_p \left| b \right|^{-k} \right\}.$$

Furthermore we denote by $I_{k,a}$ the set of all points inside $\Pi_{k,a}$. Now we define our Urysohn function $u_a$ by

$$u_a(x_1, \ldots, x_n) = \frac{1}{\kappa^n} \int_{-\frac{\kappa}{2}}^{\frac{\kappa}{2}} \cdots \int_{-\frac{\kappa}{2}}^{\frac{\kappa}{2}} \psi_a(x_1 + y_1, \ldots, x_n + y_n) \, \mathrm{d}y_1 \cdots \mathrm{d}y_n, \tag{4.2}$$

where

$$\kappa := 2c_u \left| b \right|^{-k} \tag{4.3}$$

with $c_u$ a constant and

$$\psi_a(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } (x_1, \ldots, x_n) \in I_{k,a} \\ \frac{1}{2} & \text{if } (x_1, \ldots, x_n) \in \Pi_{k,a} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $u_a$ is the desired Urysohn function which equals 1 for $z \in I_{k,a} \backslash P_{k,a}$, 0 for $z \in \mathbb{R}^n \backslash (I_{k,a} \cup P_{k,a})$, and linear interpolation in between.

We now do a Fourier transform of $u_a$ and estimate the coefficients in the same way as in Lemma 3.2 of [12].

**Lemma 4.2.** *Let* $u_a(x_1,\dots,x_n)=\sum_{(m_1,\dots,m_n)\in\mathbb{Z}^n} c_{m_1,\dots,m_n} e(m_1x_1+\cdots+m_nx_n)$ *be the Fourier series of* $u_a$. *Then the Fourier coefficients* $c_{m_1,\dots,m_n}$ *can be estimated by*

$$c_{0,\dots,0}=\frac{1}{|\mathrm{N}(b)|},\quad c_{m_1,\dots,m_n}\ll \mu^k\prod_{i=1}^n\frac{1}{r(m_i)}$$

*with*

$$r(m_i)=\begin{cases}\Delta m_i & m_i\neq 0,\\ 1 & m_i=0.\end{cases}$$

## 5. Estimation of the border

Before we proof the central proposition in the next chapter, we have to consider the error term, which mainly comes from the number of points within the tube $P_{k,a}$ defined in the previous chapter. Throughout this section we fix a positive integer $k$, which we will choose later, and a real $T$. Then we set $T_i$ as in (2.5) and define

$$F_j:=\#\left\{z\in N(\mathbf{T})\,\middle|\,\phi\left(\frac{p(z)}{b^{j+1}}\right)\in\bigcup_{a\in\mathcal{N}}P_{k,a}\mod B^{-1}\mathbb{Z}^n\right\}. \tag{5.1}$$

The main target of this section is the estimation of $F_j$.

**Proposition 5.1.** *Let* $\mu<|\mathrm{N}(b)|$ *be as in Section 4 and* $C_l$ *and* $C_u$ *be sufficiently large positive reals. Suppose that* $j$ *is a positive integer such that*

$$C_l\log\log T\le j\le d\log_{\mathrm{N}(b)}T-C_u\log\log T. \tag{5.2}$$

*Then for any positive* $\sigma_3$

$$F_j\ll \mu^kT^n\left(|\mathrm{N}(b)|^{-k}+(\log T)^{-\sigma_3}\right).$$

In order to estimate $F_j$ we apply the Erdős-Turán-Koksma Inequality.

**Lemma 5.2** ([7, Theorem 1.21])**.** *Let* $x_1,\dots,x_L$ *be points in the* $n$*-dimensional real vector space* $\mathbb{R}^n$ *and* $H$ *an arbitrary positive integer. then the discrepancy* $D_L(x_1,\dots,x_L)$ *fulfills the inequality*

$$D_L(x_1,\dots,x_L)\ll\frac{2}{H+1}+\sum_{0<\|\mathbf{h}\|_\infty\le H}\frac{1}{r(\mathbf{h})}\left|\frac{1}{L}\sum_{\ell=1}^L e(\mathbf{h}\cdot x_\ell)\right|,$$

*where* $\mathbf{h}\in\mathbb{Z}^n$ *and* $r(\mathbf{h})=\prod_{i=1}^n\max(1,|h_i|)$.

Now we are ready to prove the proposition.

*Proof of Proposition 5.1.* In order to apply the Erdős-Turán-Koksma Inequality we have to consider rectangles. Recall that the tube $P_{k,a}$ consists of rectangles as mentioned in Lemma 4.1. We split the tube $P_{k,a}$ into this family of rectangles and let $R_a$ be one of them. Then our goal is to estimate

$$F_j(R_a):=\left\{z\in N(\mathbf{T})\,\middle|\,\phi\left(\frac{p(z)}{b^{j+1}}\right)\in R_a\bmod B^{-1}\mathbb{Z}^n\right\}.$$

We set $x_z:=\phi(p(z)/b^{j+1})$ and get by the definition of the discrepancy (*cf.* [7, Definition 1.5]) that

$$F_j(R_a)\ll T^n\left(\lambda(R_a)+D_L(\{x_z\})\right), \tag{5.3}$$

where $L$ is the number of elements in $N(\mathbf{T})$ and $T$ is the geometric mean of the $T_i$. Thus we get by Lemma 3.3 that

$$L=\frac{2^r\pi^{r_2}}{\sqrt{D}}T^n+\mathcal{O}\left(T^{n-1}\right). \tag{5.4}$$

In order to estimate the discrepancy we use Lemma 5.2 to get

$$D_L(\{x_z\}) \ll \frac{2}{H+1} + \sum_{0<\|\mathbf{h}\|_\infty \leq H} \frac{1}{r(\mathbf{h})} \left| \frac{1}{L} \sum_{z\in N(\mathbf{T})} e(\mathbf{h}\cdot x_z) \right|. \tag{5.5}$$

Our aim is the application of Proposition 3.2. Since $E$ is defined in (2.2) as $E = e \circ \mathrm{Tr}$ we have to rewrite the exponential sum with scalar multiplication into one involving the trace. It is easy to see that the following function suffices our purpose.

$$\tau(z) := (\mathrm{Tr}(z), \mathrm{Tr}(bz), \dots, \mathrm{Tr}(b^{n-1}z)) = \Xi\phi(z), \tag{5.6}$$

where $\Xi = VV^T$ and $V$ is the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ b & b^{(2)} & \cdots & (b^{(n)})^{n-1} \\ \vdots & \vdots & & \vdots \\ b^{n-1} & (b^{(2)})^{n-1} & \cdots & (b^{(n)})^{n-1} \end{pmatrix}.$$

Thus we get

$$\mathbf{h}\cdot x_z = \mathbf{h}\cdot\phi\left(\frac{p(z)}{b^{j+1}}\right) = \mathbf{h}\Xi^{-1}\tau\left(\frac{p(z)}{b^{j+1}}\right) = \mathrm{Tr}\left(\sum_{i=1}^n \tilde{h_i} b^{i-1}\frac{p(z)}{b^{j+1}}\right), \tag{5.7}$$

where we have set $(\tilde{h_1}, \tilde{h_2}, \dots, \tilde{h_n}) := \mathbf{h}\Xi^{-1}$.

Plugging (5.4), (5.5) and (5.7) into (5.3) and applying Lemma 4.1 yields

$$F_j(R_a) \ll \lambda(R_a) T^n + \frac{T^n}{H+1} + \sum_{0<\|\mathbf{h}\|_\infty \leq H} \frac{1}{r(\mathbf{h})} \sum_{z\in N(\mathbf{T})} E\left(\sum_{i=1}^n \tilde{h_i} b^{i-1}\frac{p(z)}{b^{j+1}}\right), \tag{5.8}$$

where

$$H = (\log T)^{\sigma_4}. \tag{5.9}$$

Now we want to apply Proposition 3.2 for the last sum. Since $\sum_{i=1}^n \tilde{h_i} b^{i-1} p(z)$ is a polynomial of degree $d$ we write $\xi$ for its leading coefficient for short. Then we apply Lemma 3.1 to get $a \in \delta^{-1}$ and $q \in \mathcal{O}_K$ such that

$$\left|\frac{\xi^{(i)}}{(b^{(i)})^{j+1}} q^{(i)} - a^{(i)}\right| \leq T_i^{-d}(\log T)^{\sigma_1} \quad \text{and} \quad 0 < q^{(i)} \leq T_i^d (\log T)^{-\sigma_1} \quad \text{for} \quad 1 \leq i \leq r.$$

Now we distinguish three cases according to the size of $\lceil q \rceil$ (defined in (2.1)).

- **Case 1,** $\lceil q \rceil \geq (\log T)^{\sigma_1}$: By Proposition 3.2 we get

$$\sum_{z\in N(\mathbf{T})} E\left(\sum_{i=1}^n \tilde{h_i} b^{i-1}\frac{p(z)}{b^{j+1}}\right) \ll T^n (\log T)^{-\sigma_0}.$$

- **Case 2,** $2 \leq \lceil q \rceil < (\log T)^{\sigma_1}$: Let $1 \leq i \leq n$ be such that $\left|q^{(i)}\right| \geq 2$. Then by noting $\left|q^{(i)}\right| \leq \lceil q \rceil$ we get

$$\left|\frac{\xi^{(i)}}{(b^{(i)})^{j+1}} q^{(i)} - a^{(i)}\right| < \frac{(\log T)^{\sigma_1}}{T_i^d} \leq \frac{1}{\left|q^{(i)}\right|}$$

and thus (using our first successive minimum $\lambda_1$)

$$\left|\frac{\xi^{(i)}}{(b^{(i)})^{j+1}}\right| \geq \left|\frac{a^{(i)}}{q^{(i)}}\right| - \frac{1}{\left|q^{(i)}\right|^2} \geq \lambda_1\left(\frac{1}{\left|q^{(i)}\right|} - \frac{1}{\left|q^{(i)}\right|^2}\right) \geq \lambda_1 \frac{1}{2\left|q^{(i)}\right|} \gg (\log T)^{-\sigma_1}. \tag{5.10}$$

Therefore by noting Lemma 2.1 and our definition of $H$ in (5.9) we have

$$\left|b^{(i)}\right|^{j+1} \ll \left|\xi^{(i)}\right| (\log T)^{\sigma_1} \ll nH(\log T)^{\sigma_1} \ll (\log T)^{\sigma_1+\sigma_4},$$

which yields

$$j \ll \frac{(\sigma_1+\sigma_2)}{\log\left|b^{(i)}\right|}\log\log T$$

contradicting the lower bound for $j$ in (5.2) for sufficiently large $C_l$.

- **Case 3,** $0 < \lceil q \rceil < 2$: We have to consider two further cases according to whether $a \neq 0$ or $a = 0$.
  - **Case 3.1,** $\left\lceil \frac{\xi}{b^{j+1}} q \right\rceil \geq \frac{\lambda_1}{2}$: Let $1 \leq i \leq n$ be such that $\left|\frac{\xi^{(i)}}{(b^{(i)})^{j+1}} q^{(i)}\right| \geq \frac{\lambda_1}{2}$. Then we get with $H$ as in (5.9)

$$\left|b^{(i)}\right|^{j+1} \ll H = (\log T)^{\sigma_4}$$

    which again contradicts the lower bound of $j$ in (5.2).
  - **Case 3.2,** $\left\lceil \frac{\xi}{b^{j+1}} q \right\rceil < \frac{\lambda_1}{2}$: Since $\lambda_1$ is the first successive minima for the sphere with respect to the lattice of $\delta^{-1}$ we get that $a = 0$. Thus for $1 \leq i \leq n$

$$\left|\frac{\xi^{(i)}}{(b^{(i)})^{j+1}} q^{(i)}\right| < \frac{(\log T)^{\sigma_1}}{T_i^d}.$$

    Taking the norm we get

$$\mathrm{N}\left(\frac{\xi}{b^{j+1}} q\right) = \frac{\mathrm{N}(\xi)}{\mathrm{N}(b)^{j+1}} \mathrm{N}(q) < \frac{(\log T)^{n\sigma_1}}{T^{nd}}.$$

    This implies

$$j + 1 \geq nd \log_{|\mathrm{N}(b)|} T - \frac{n\sigma}{\log|\mathrm{N}(b)|} \log\log T + C$$

    contradicting the upper bound of $j$ in (5.2) for sufficiently large $C_u$.

Thus we get for all $j$ satisfying (5.2), that

$$\sum_{z \in N(\mathbf{T})} E\left(\sum_{i=1}^n \tilde{h_i} b^{i-1} \frac{f(z)}{b^{j+1}}\right) \ll T^n (\log T)^{-\sigma_0}.$$

Plugging this into (5.8) together with (5.9) yields

$$\begin{aligned} F_j(R_a) &\ll T^n \lambda(R_a) + \frac{T^n}{(\log T)^{\sigma_4}} + T^n (\log T)^{-\sigma_0} \sum_{0 < \|\mathbf{h}\|_\infty \leq H} \frac{1}{r(\mathbf{h})} \\ &\ll T^n \lambda(R_a) + \frac{T^n}{(\log T)^{\sigma_4}} + T^n (\log T)^{-\sigma_0} (\log\log T)^n. \end{aligned}$$

Finally setting $\sigma_4 := \sigma_0/2$ and summing over all rectangles $R_a$ yields

$$F_j \ll \mu^k T^n \left(|\mathrm{N}(b)|^{-k} + (\log T)^{-\sigma_0/2}\right).$$

By setting $\sigma_3 = \sigma_0/2$ the proposition is proven. □

## 6. The central proposition

In this section we want to state and proof the central proposition. It will count the number of elements with given digits in the expansion. This will provide us with the main tool in order to proof Theorem 2.3.

**Proposition 6.1.** *Let $T \geq 0$ and $T_i$ for $1 \leq i \leq n$ be defined as in (2.5). Let $L$ be the maximum length of the $b$-adic expansion of $z \in N(\mathbf{T})$ and let $C_l$ and $C_u$ be sufficiently large. Then for*

$$C_l \log L \leq l_1 < l_2 < \cdots < l_h \leq dL - C_u \log L \tag{6.1}$$

*we have,*

$$\Theta := \#\left\{z \in N(\mathbf{T}) \middle| a_{l_j}(f(z)) = b_j, j = 1, \ldots, h\right\} = \frac{2^r \pi^{r_2}}{\sqrt{D}\,|\mathrm{N}(b)|^h} T^n + \mathcal{O}\left(T^n (\log T)^{-\sigma_0}\right)$$

*uniformly for $T \to \infty$, where $b_j \in \mathcal{N}$ are given and $\sigma_0$ is an arbitrary positive constant.*

*Proof.* We recall our Urysohn function $u_a$ (defined in (4.2)) and set for $\nu\in\mathbb{R}^n$

$$t(\nu)=u_{b_1}(B^{-l_1-1}\nu)\cdots u_{b_h}(B^{-l_h-1}\nu),$$

where $B$ is the matrix defined in (4.1).

Now we want to apply the Fourier transformation, which we developed in Lemma 4.2. Therefore we set

$$\mathcal{M}:=\left\{M=(\mu_1,\dots,\mu_h)|\mu_j=(m_{j1},\dots,m_{jn})\in\mathbb{Z}^n,\text{ for }j=1,\dots,h\right\}.$$

An application of Lemma 4.2 yields

$$t(\nu)=\sum_{M\in\mathcal{M}}T_M e\left(\sum_{j=1}^h\mu_jB^{-l_j-1}\nu\right),$$

where $T_M=\prod_{j=1}^h c_{m_{j1},\dots,m_{jn}}$. Combining these results we get

$$\left|\Theta-\sum_{z\in N(\mathbf{T})}t(\phi(p(z)))\right|\leq F_{l_1}+\cdots+F_{l_h}. \tag{6.2}$$

Using the function $\tau$ defined in (5.6) we get

$$\sum_{z\in N(\mathbf{T})}t(\phi(p(z)))=\sum_{M\in\mathcal{M}}T_M\sum_{z\in N(\mathbf{T})}e\left(\sum_{j=1}^h\mu_jB^{-l_j-1}\Xi^{-1}\tau(p(z))\right).$$

Setting

$$\tilde{\mu_j}=(\tilde{m_{j1}},\dots,\tilde{m_{jn}}):=\mu_jB^{-l_j-1}\Xi^{-1}\quad(j=1,\dots,h)$$

yields

$$\sum_{z\in N(\mathrm{T})}t(\phi(p(z)))=\sum_{M\in\mathcal{M}}T_M\sum_{z\in N(\mathbf{T})}E\left(\sum_{j=1}^h\sum_{i=1}^n\frac{\tilde{m_{ji}}p(z))}{b^{l_j+1}}\right).$$

We want to apply the same ideas as in the proof of Proposition 5.1. For $1\leq j\leq h$ we set $\xi_j$ to be the leading coefficient of $\sum_{i=1}^n\tilde{m_{ji}}p(z)$. Then we apply Lemma 3.1 with $N_i=T_i^d(\log T)^{-\sigma_1}$ for $1\leq i\leq r$ in order to get that there exist $a\in\delta^{-1}$ and $q\in\mathcal{O}_K$ such that

$$\left|\sum_{j=1}^h\frac{\xi_j^{(i)}}{(b^{(i)})^{l_j+1}}q^{(i)}-a^{(i)}\right|<\frac{(\log T)^{\sigma_1}}{T_i^d}\quad\text{and}\quad 0<\left|q^{(i)}\right|<\frac{T_i^d}{(\log T)^{\sigma_1}}\quad\text{for}\quad 1\leq i\leq n.$$

Now we again distinguish several cases.

- **Case 1,** $\lceil q\rceil\geq(\log T)^{\sigma_1}$: We apply Proposition 3.2 and get

$$\sum_{z\in M(\mathbf{T})}E\left(\sum_{j=1}^h\sum_{i=1}^n\frac{\tilde{m_{ji}}p(z))}{b^{l_j+1}}\right)\ll T^n(\log T)^{-\sigma_0}.$$

- **Case 2,** $2\leq\lceil q\rceil<(\log T)^{\sigma_1}$: In the same manner as in (5.10) we get

$$(\log T)^{\sigma_1}\leq\left|\sum_{j=1}^h\frac{\xi_j^{(i)}}{(b^{(i)})^{l_j+1}}\right|\leq\frac{\sum_{j=1}^h\left|\xi_j^{(i)}\right|}{\left|b^{(i)}\right|^{l_1+1}},$$
$$l_1+1\ll\log\log T.$$

  Together with the definition of $L$ this contradicts the lower bound of $l_1$ for sufficiently large $C_l$ in (6.1).
- **Case 3,** $0<\lceil q\rceil<2$: We have to consider two sub cases

– **Case 3.1,** $\left|\sum_{j=1}^{h}\frac{\xi_j}{b^{l_j+1}}q\right| \geq \frac{\lambda_1}{2}$: Let $1\leq i\leq n$ be such that

$$\frac{\lambda_1}{2} \leq \left|\sum_{j=1}^{h}\frac{\xi_j^{(i)}}{(b^{(i)})^{l_j+1}}q^{(i)}\right| \leq \frac{\sum_{j=1}^{h}\left|\xi_j^{(i)}\right|}{\left|b^{(i)}\right|^{l_1+1}}\left|q^{(i)}\right|,$$

then

$$l_1+1 \ll \log\log T$$

again contradicts the lower bound of $l_1$ for sufficiently large $C_l$ in (6.1).

– **Case 3.2,** $\left|\sum_{j=1}^{h}\frac{\xi_j}{b^{l_j+1}}q\right| < \frac{\lambda_1}{2}$: By Minkowski's theorem (*cf.* [15]) we get that $a=0$. Thus for $1\leq i\leq n$

$$\left|\sum_{j=1}^{h}\frac{\xi_j^{(i)}}{(b^{(i)})^{l_j+1}}q^{(i)}\right| = \left|\frac{1}{(b^{(i)})^{l_h+1}}\sum_{j=1}^{h}\xi_j^{(i)}(b^{(i)})^{l_h-l_j}q^{(i)}\right| \leq \frac{(\log T)^{\sigma_1}}{T_i^d}$$

which implies (taking the norm of the left side)

$$l_h+1 \geq nd\log_{|\mathrm{N}(b)|}T - c(\log\log_{|\mathrm{N}(b)|}T)$$

contradicting the upper bound for sufficiently large $C_u$.

After these considerations it is clear, that **Case 1** is the only possible case which suffices the hypotheses in (6.1). Plugging this into (6.2) and applying Lemma 4.2 and Lemma 3.3 for the coefficient $c_{0,\ldots,0}$ yields

$$\Theta = \frac{2^r\pi^{r_2}}{\sqrt{D}\,|\mathrm{N}(b)|^h}T^n + \mathcal{O}\left(T^n(\log T)^{-\sigma_0}\sum_{0\neq M\in\mathcal{M}}T_M\right) + \mathcal{O}\left(\sum_{j=1}^{h}F_j\right).$$

An application of Proposition 5.1 with $k\ll\log\log T$ and the observation that

$$\sum_{M\in\mathcal{M}}|T_M| \ll \kappa^{-2h} \ll |b|^{2hk} \ll (\log T)^{\sigma_0/2},$$

where we used the definition of $\kappa$ in (4.3), proves the proposition. □

## 7. Proof of Theorem 2.3

At this point we will need the Fréchet-Shohat Theorem which we state for completeness.

**Lemma 7.1** ([10, Lemma 1.43]). *Let $(F_n(z))_{n\geq1}$ be a sequence of distribution functions. For each non-negative integer $k$ let*

$$\alpha_k = \lim_{n\to\infty}\int_{-\infty}^{\infty}z^k\mathrm{d}F_n(z)$$

*exist.*

*Then there is a subsequence of $F_{n_j}(z)$ $(n_1<n_2<\cdots)$, which converges weakly to a limiting distribution $F(z)$ for which*

$$\alpha_k = \int_{-\infty}^{\infty}z^k\mathrm{d}F(z) \quad (k=0,1,\ldots).$$

*Moreover, if the set of moments $\alpha_k$ determine $F(z)$ uniquely, then as $n\to\infty$ the distribution $F_n(z)$ converge weakly to $F(z)$.*

For the proof of Theorem 2.3 we mainly follow the proof of Theorem 1.2 of Gittenberger and Thuswaldner [12]. In the same manner as in their proof we set $C:=\max(C_l,C_u)$, $A:=[C\log L]$ and $B:=L-A$, where $L$, $C_l$ and $C_u$ are defined in the statement of Proposition 6.1. Furthermore we define the truncated function $f'$ to be

$$f'(p(z)) = \sum_{j=A}^{B}f(a_j(p(z))b^j).$$

By the definition of $A$ and $f(ab^j) \ll 1$ with $a \in \mathcal{N}$ we get that $f'(p(z)) = f(p(z)) + \mathcal{O}(\log L)$. In the same manner we define the truncated mean and standard deviation

$$M_b'(T) := \sum_{j=A}^{B} m_j \quad \text{and} \quad D_b'^2(T) := \sum_{j=A}^{B} \sigma_j^2.$$

Since $M_b(T) - M_b'(T) = \mathcal{O}(\log L)$ and $D_b^2(T) - D_b'^2(T) = \mathcal{O}(\log L)$ we get that it suffices to show that

$$\frac{1}{\#N(\mathbf{T})} \# \left\{ z \in N(\mathbf{T}) \middle| \frac{f'(p(z)) - M_b'(T^d)}{D_b'(T^d)} < y \right\} \longrightarrow \Phi(y).$$

By Lemma 7.1 this holds true if and only if the moments

$$\xi_k(T) := \frac{1}{\#N(\mathbf{T})} \sum_{z \in N(T)} \left( \frac{f'(p(z)) - M_b'(T^d)}{D_b'(T^d)} \right)^k$$

converge to the moments of the normal law for $T \to \infty$. We will show the last statement by comparing the moments $\xi_k$ with

$$\eta_k(T) := \frac{1}{\#N(\mathbf{T}^d)} \sum_{z \in N(T^d)} \left( \frac{f'(z) - M_b'(T^d)}{D_b'(T^d)} \right)^k,$$

where $\mathbf{T}^d = (T_1^d, \ldots, T_r^d)$.

An application of Proposition 6.1 gives that

$$\xi_k(T) - \eta_k(T) \to 0 \quad \text{for} \quad T \to \infty.$$

Furthermore we get by Lemma 2.2 that these sums consist of independently identically distributed random variables (with possible $2C$ exceptions). By the central limit theorem we get that their distribution converges to the normal law. Thus the $\eta_k(T)$ converge to the moments of the normal law. This yields

$$\lim_{T \to \infty} \xi_k(T) = \lim_{T \to \infty} \eta_k(T) = \int x^k \mathrm{d}\Phi.$$

We apply the Fréchet-Shohat Theorem again to prove the theorem.

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## References

[1] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, and J. M. Thuswaldner, *Generalized radix representations and dynamical systems. I*, Acta Math. Hungar. **108** (2005), no. 3, 207–238.

[2] S. Akiyama, H. Brunotte, and A. Pethő, *Cubic CNS polynomials, notes on a conjecture of W. J. Gilbert*, J. Math. Anal. Appl. **281** (2003), no. 1, 402–415.

[3] N. L. Bassily and I. Kátai, *Distribution of the values of q-additive functions on polynomial sequences*, Acta Math. Hungar. **68** (1995), no. 4, 353–361.

[4] H. Brunotte, A. Huszti, and A. Pethő, *Bases of canonical number systems in quartic algebraic number fields*, J. Théor. Nombres Bordeaux **18** (2006), no. 3, 537–557.

[5] H. Delange, *Sur la fonction sommatoire de la fonction"somme des chiffres"*, Enseignement Math. (2) **21** (1975), no. 1, 31–47.

[6] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, J. Théor. Nombres Bordeaux **10** (1998), no. 1, 17–32.

[7] M. Drmota and R. F. Tichy, *Sequences, discrepancies and applications*, Lecture Notes in Mathematics, vol. 1651, Springer-Verlag, Berlin, 1997.

[8] J. M. Dumont, P. J. Grabner, and A. Thomas, *Distribution of the digits in the expansions of rational integers in algebraic bases*, Acta Sci. Math. (Szeged) **65** (1999), no. 3-4, 469–492.

[9] J. M. Dumont and A. Thomas, *Gaussian asymptotic properties of the sum-of-digits function*, J. Number Theory **62** (1997), no. 1, 19–38.

[10] P. D. T. A. Elliott, *Probabilistic number theory. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 239, Springer-Verlag, New York, 1979, Mean-value theorems.
[11] B. Gittenberger and J. M. Thuswaldner, *The moments of the sum-of-digits function in number fields*, Canad. Math. Bull. **42** (1999), no. 1, 68–77.
[12] ———, *Asymptotic normality of b-additive functions on polynomial sequences in the Gaussian number field*, J. Number Theory **84** (2000), no. 2, 317–341.
[13] P. J. Grabner, P. Kirschenhofer, and H. Prodinger, *The sum-of-digits function for complex bases*, J. London Math. Soc. (2) **57** (1998), no. 1, 20–40.
[14] L.-K. Hua, *Additive Primzahltheorie*, B. G. Teubner Verlagsgesellschaft, Leipzig, 1959.
[15] ———, *Introduction to number theory*, Springer-Verlag, Berlin, 1982, Translated from the Chinese by Peter Shiu.
[16] I. Kátai and I. Környei, *On number systems in algebraic number fields*, Publ. Math. Debrecen **41** (1992), no. 3-4, 289–294.
[17] I. Kátai and B. Kovács, *Kanonische Zahlensysteme in der Theorie der quadratischen algebraischen Zahlen*, Acta Sci. Math. (Szeged) **42** (1980), no. 1-2, 99–107.
[18] ———, *Canonical number systems in imaginary quadratic fields*, Acta Math. Acad. Sci. Hungar. **37** (1981), no. 1-3, 159–164.
[19] I. Kátai and P. Liardet, *Additive functions with respect to expansions over the set of Gaussian integers*, Acta Arith. **99** (2001), no. 2, 173–182.
[20] B. Kovács, *Canonical number systems in algebraic number fields*, Acta Math. Acad. Sci. Hungar. **37** (1981), no. 4, 405–407.
[21] B. Kovács and A. Pethő, *Number systems in integral domains, especially in orders of algebraic number fields*, Acta Sci. Math. (Szeged) **55** (1991), no. 3-4, 287–299.
[22] ———, *On a representation of algebraic integers*, Studia Sci. Math. Hungar. **27** (1992), no. 1-2, 169–172.
[23] M. G. Madritsch, *A note on normal numbers in matrix number systems*, Math. Pannon. **18** (2007), no. 2, 219–227.
[24] ———, *Generating normal numbers over Gaussian integers*, Acta Arith. **135** (2008), no. 1, 63–90.
[25] C. Mauduit and A. Sárközy, *On the arithmetic structure of the integers whose sum of digits is fixed*, Acta Arith. **81** (1997), no. 2, 145–173.
[26] T. Mitsui, *On the Goldbach problem in an algebraic number field. I, II*, J. Math. Soc. Japan **12** (1960), 290–324, 325–372.
[27] W. Müller, J. M. Thuswaldner, and R. F. Tichy, *Fractal properties of number systems*, Period. Math. Hungar. **42** (2001), no. 1-2, 51–68.
[28] A. Pethő and R. F. Tichy, *S-unit equations, linear recurrences and digit expansions*, Publ. Math. Debrecen **42** (1993), no. 1-2, 145–154.
[29] P. Ribenboim, *Classical theory of algebraic numbers*, Universitext, Springer-Verlag, New York, 2001.
[30] K. Scheicher and J. M. Thuswaldner, *Canonical number systems, counting automata and fractals*, Math. Proc. Cambridge Philos. Soc. **133** (2002), no. 1, 163–182.
[31] C. L. Siegel, *Generalization of Waring's problem to algebraic number fields*, Amer. J. Math. **66** (1944), 122–136.
[32] J. M. Thuswaldner, *The sum of digits function in number fields*, Bull. London Math. Soc. **30** (1998), no. 1, 37–45.
[33] ———, *The sum of digits function in number fields: distribution in residue classes*, J. Number Theory **74** (1999), no. 1, 111–125.
[34] J. M. Thuswaldner and R. F. Tichy, *Waring's problem with digital restrictions*, Israel J. Math. **149** (2005), 317–344, Probability in mathematics.
[35] Y. Wang, *Diophantine equations and inequalities in algebraic number fields*, Springer-Verlag, Berlin, 1991.

(M. G. Madritsch) Institute of Statistics, Graz University of Technology, A-8010 Graz, Austria
*E-mail address*: madritsch@finanz.math.tugraz.at