A NOTE ON GENERALIZED RADIX REPRESENTATIONS AND DYNAMICAL SYSTEMS

MANFRED G. MADRITSCH AND ATTILA PETHŐ

Abstract. Akiyama *et al.* [2] proved an asymptotic formula for the distribution of CNS polynomials with fixed constant term. The objective of the present paper is to improve that result by providing an error term too.

1. Introduction

Let $d \geq 1$ be an integer. To each $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ we associate a mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ by setting for $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$.

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_d, -\lfloor \mathbf{r} \cdot \mathbf{a} \rfloor),$$

where $\mathbf{r} \cdot \mathbf{a} = a_1 r_1 + \cdots + a_d r_d$ denotes the inner product of the vectors $\mathbf{r}$ and $\mathbf{a}$. We call $\tau_{\mathbf{r}}$ a shift radix system (SRS for short) if for each $\mathbf{a} \in \mathbb{Z}^d$ there exists some $k > 0$ such that $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$. These systems were introduced in 2005 by Akiyama et al. [1] and they turned out to be generalizations of several notions of well-known number systems. For certain parameters $\mathbf{r}$ SRS are related to β-expansion having a certain finiteness property (F) (*cf.* [8, 10, 13, 14]) or to canonical number systems (*cf.* [9, 11] and [1, 4] for the connection with SRS).

In the present paper we will only concentrate on those elements $\mathbf{r} \in \mathbb{R}^d$ such that $\tau_{\mathbf{r}}$ is ultimately periodic for all $\mathbf{a} \in \mathbb{Z}^d$. Thus for $d \geq 1$ an integer let

$$\mathcal{D}_d := \left\{ \mathbf{r} \in \mathbb{R}^d : \ \forall \mathbf{a} \in \mathbb{Z}^d \text{ the sequence } (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic} \right\}.$$

The elements of $\mathcal{D}_d$ are in strong relation with the set of contracting polynomials. In particular, we define $\mathcal{E}_d(r)$ to be the set of all monic polynomials having spectral radius less than r, *i.e.*,

$$\mathcal{E}_d(r) := \left\{ (r_1, \ldots, r_d) \in \mathbb{R}^d : \ X^d + r_d X^{d-1} + \cdots + r_1 \text{ has only roots in } y \in \mathbb{C} \text{ with } |y| < r \right\}.$$

If $r = 1$ then we set $\mathcal{E}_d := \mathcal{E}_d(1)$ for short.

The set $\mathcal{E}_d = \mathcal{E}_d(1)$ was characterized by Schur [15] as

$$\mathcal{E}_d = \left\{ (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d \middle| \forall k \in \{0, \ldots, d-1\} \text{ we have } \det(\delta_k(r_0, \ldots, r_{d-1})) > 0 \right\},$$

where $\delta_k(r_0, \ldots, r_{d-1}$ is the $2(k+1) \times 2(k+1)$-matrix defined by

$$\delta_k(r_0, \ldots, r_{d-1}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & r_0 & \cdots & \cdots & r_k \\ r_{d-1} & \ddots & \ddots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ r_{d-k-1} & \cdots & r_{d-1} & 1 & 0 & \cdots & 0 & r_0 \\ r_0 & 0 & \cdots & 0 & 1 & r_{d-1} & \cdots & r_{d-k-1} \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & \vdots & \ddots & \ddots & r_{d-1} \\ r_k & \cdots & \cdots & r_0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

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Furthermore Fam and Meditch [6] showed that the set $\mathcal{E}_d$ is simply connected and for $d \geq 2$ it is bounded by three hypersurfaces two of which are hyperplanes. Finally we note that $\mathcal{E}_{d-1}(r)$ is defined by similar means.

Now we want to link the set $\mathcal{E}$ with the sets of Pisot, Salem and canonical polynomials, respectively. We start with the relation to Pisot and Salem polynomials. To this end let $P(X) = X^d - b_1X^{d-1} - \cdots - b_d \in \mathbb{Z}[X]$ be an irreducible polynomial over $\mathbb{Z}$.

- If P has a real root greater than one and all other roots are located in the open unit disk, then P is called a Pisot polynomial.
- If P has a real root greater than one and all other roots are located in the closed unit disk and at least one of them has modulus 1, then P is called a Salem polynomial.

Then we define for each $d \in \mathbb{N}$, $d \geq 1$ and each $M \in \mathbb{N}$ the sets

$$\begin{aligned}\mathcal{B}_d &:= \left\{(b_1,\dots,b_d) \in \mathbb{Z}^d :\ X^d - b_1X^{d-1} - \cdots - b_d \text{ is Pisot or Salem polynomial}\right\},\\ \mathcal{B}_d(M) &:= \left\{(b_2,\dots,b_d) \in \mathbb{Z}^{d-1} : (M,b_2,\dots,b_d) \in \mathcal{B}_d\right\}.\end{aligned}$$

With these notations Akiyama *et al.* [3] were able to show the following

Theorem 1.1 ([3, Theorem 1.2]). *Let $d \geq 2$. Then*

$$\left|\frac{|\mathcal{B}_d(M)|}{M^{d-1}} - \lambda_{d-1}(\mathcal{D}_{d-1})\right| \ll M^{-\frac{1}{d-1}},$$

where λ_{d-1} denotes the $(d-1)$-dimensional Lebesgue measure.

Now we concentrate on the relation of $\mathcal{D}_d$ and canonical number systems. Therefore let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0 \in \mathbb{Z}[X]$ be a monic polynomial of degree d with $p_0 \geq 2$, and set $\mathcal{N} = \{0,1,\dots,p_0-1\}$. Furthermore we denote by x the image of X under the canonical epimorphism from $\mathbb{Z}[X]$ to $R := \mathbb{Z}[X]/(P(X)\mathbb{Z}[X])$. Since P is monic it is clear that every element $A(X)$ of R has a unique representation of degree at most $d-1$, say

$$A(X) = A_{d-1}X^{d-1} + \cdots + A_1X + A_0 \quad (A_i \in \mathbb{Z}).$$

Now we want to analyze if every element has a representation to base X having digits in $\mathcal{N}$. Therefore let $\mathcal{G} := \{A(X) \in \mathbb{Z}[X] : \deg A < d\}$ be the set of all elements of degree less than d and

$$T_P(A) = \sum_{i=0}^{d-1}(A_{i+1} - qp_{i+1})X^i,$$

where $A_d = 0$ and $q = \lfloor A_0/p_0 \rfloor$, be the "division map". Then clearly $T_p : \mathcal{G} \to \mathcal{G}$ and

$$A(x) = (A_0 - qp_0) + xT_P(A),$$

where $A_0 - qp_0 \in \mathcal{N}$. Thus this provides our desired representation. If for each $A \in \mathcal{G}$ there is a $k \in \mathbb{N}$ such that $T_P^k(A) = 0$, then we call P a canonical number system polynomial (CNS polynomial for short).

In order to draw the connection of CNS polynomials and SRS we define for each $d \geq 1$ and $M \geq 1$ integers the sets

$$\begin{aligned}\mathcal{C}_d &:= \left\{(p_0,\dots,p_{d-1}) \in \mathbb{Z}^d :\ |p_0| \geq 2 \text{ and } T_{X^d+p_{d-1}X^{d-1}+\cdots+p_0} \text{ has only finite orbits}\right\}\\ \mathcal{C}_d(M) &:= \{(p_1,\dots,p_{d-1} : (M,p_1,\dots,p_{d-1}) \in \mathcal{C}_d\}.\end{aligned}$$

The connection between $\mathcal{C}_d$ and $\mathcal{D}_d$ was proven in the first part of a series of papers by Akiyama *et al.* [1]. In particular, they proved that

$$(p_0,p_1,\dots,p_{d-1}) \in \mathcal{C}_d \quad \text{if and only if} \quad \left(\frac{1}{p_0},\frac{p_{d-1}}{p_0},\dots,\frac{p_1}{p_0}\right) \in \mathcal{D}_d.$$

In the third part of that series of papers Akiyama *et al.* [2] proved an asymptotic formula for the cardinality of $\mathcal{C}_d(M)$. More precisely they proved

Theorem 1.2 ([2, Theorem 5.1])**.** *Let $d \geq 2$ be a positive integer. Then*

$$\lim_{M\to\infty} \frac{|\mathcal{C}_d(M)|}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_{d-1}).$$

The objective of the present paper is to improve this result. Combining methods originating from the proofs of Theorems 1.1 and 1.2 we are able to estimate the speed of convergence too. More precisely we prove

Theorem 1.3. *Let $d \geq 2$ be a positive integer. Then*

$$\left| \frac{|\mathcal{C}_d(M)|}{M^{d-1}} - \lambda_{d-1}(\mathcal{D}_{d-1}) \right| \ll M^{-\frac{1}{d-1}}.$$

In [1, Lemma 4.1, 4.2 and 4.3] it was proved that

$$\operatorname{int}(\mathcal{D}_d) = \mathcal{E}_d.$$

The structure of $\mathcal{E}_d$ and its Lebesgue measure has been analyzed by Kirschenhofer *et al.* [12]. Using a result by Fam [7] together with Barnes G-function they calculated that

$$\lambda_d(\mathcal{D}_d) = \lambda_d(\mathcal{E}_d) = \begin{cases} \frac{2^{2n^2+n}\Gamma(n+1)G(n+1)^4}{G(2n+2)} & \text{if } d = 2n, \\ \frac{2^{2n^2+3n+1}G(n+2)^4}{\Gamma(n+1)G(2n+2)} & \text{if } d = 2n+1, \end{cases}$$

We note that for positive integers the Barnes G-function equals the superfactorials, *i.e.*, $G(n+2) = \prod_{j=1}^{n} j!$ for $n \in \mathbb{N}$.

2. Auxiliary lemmata

Let $d \geq 1$ be a positive integer. Then for $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ we denote by $\rho(\mathbf{r})$ the spectral radius of the polynomial $P(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$, *i.e.*,

$$\rho(\mathbf{r}) := \rho(P) = \max\{|\alpha| : P(\alpha) = 0\}.$$

Our first tool deals with the relation of the spectral radius if we change the coefficients of the polynomial a little bit.

Lemma 2.1 ([3, Lemma 4.1])**.** *Let $d \in \mathbb{N}$ and $\rho, \varepsilon > 0$. Then there exists a constant $c > 0$ depending only on d and ρ with the following property: if all roots $\alpha \in \mathbb{C}$ of the polynomial $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0 \in \mathbb{R}[X]$ satisfy $|\alpha| < \rho$ and $Q(X) = X^d + q_{d-1}X^{d-1} + \cdots + q_0 \in \mathbb{R}[X]$ is chosen such that $|p_i - q_i| < \varepsilon$, $i = 0, \ldots, d-1$ then for each root β of $Q(X)$ there exists a root α of $P(X)$ satisfying*

$$|\beta - \alpha| < c\varepsilon^{\frac{1}{d}}.$$

In particular, all roots β of $Q(X)$ satisfy $|\beta| < \rho + c\varepsilon^{\frac{1}{d}}$.

In the proof we will approximate $\mathcal{D}_d$ by polynomials having larger and smaller spectral radius. Therefore we need estimates of the Lebesgue measure of the difference sets.

Lemma 2.2 ([3, Lemma 4.2])**.** *Let $0 < \eta < 1$. Then we have*

$$\lambda_d\left(\mathcal{E}_d(1+\eta) \setminus \mathcal{D}_d\right) \leq 2^{d(d+1)/2}\lambda_d(\mathcal{E}_d)\eta$$

and

$$\lambda_d\left(\mathcal{D}_d \setminus \mathcal{E}_d(1-\eta)\right) \leq 2^{d(d+1)/2}\lambda_d(\mathcal{E}_d)\eta.$$

The central tool is an estimation of the integral points in a bounded region which is due to H. Davenport.

Lemma 2.3 ([5, Theorem])**.** *Let $\mathcal{R}$ be a closed bounded region in the n dimensional space $\mathbb{R}^n$ and let $\mathrm{N}(\mathcal{R})$ and $\mathrm{V}(\mathcal{R})$ denote the number of points with integral coordinates in $\mathcal{R}$ and the volume of $\mathcal{R}$, respectively. Suppose that:*

- *Any line parallel to one of the n coordinate axes intersects $\mathcal{R}$ in a set of points which, if not empty, consists of at most h intervals.*

- *The same is true (with m in place of n) for any of the m dimensional regions obtained by projecting $\mathcal{R}$ on one of the coordinate spaces defined by equating a selection of $n-m$ of the coordinates to zero; and this condition is satisfied for all m from 1 to $n-1$.*

Then

$$|\mathrm{N}(\mathcal{R})-\mathrm{V}(\mathcal{R})|\leq\sum_{m=0}^{n-1}h^{n-m}V_m,$$

where V_m is the sum of the m dimensional volumes of the projections of $\mathcal{R}$ on the various coordinate spaces obtained by equating any $n-m$ coordinates to zero, and $V_0=1$ by convention.

3. PROOF OF THEOREM 1.3

The proof consists of mainly two steps. First we cover $\mathcal{D}_d$ by hypercubes. At this step we have to show that the intersection of two such hypercubes does not have big measure. Then, in the second step, we will count the number of hypercubes in $\mathbb{R}^d$ by providing a covering for the border of $\mathcal{D}_d$.

Since we are only interested in an asymptotic, throughout the proof we will denote by c an arbitrary constant. This constant might not be the same in different occurrences. However, if the reader is awake, there will be no problem.

Thus we start with the embedding of $\mathcal{C}_d$ in $\mathbb{R}^{d-1}$:

$$\psi:\mathcal{C}_d\to\mathbb{R}^{d-1},\quad(p_0,\ldots,p_{d-1})\mapsto\left(\frac{p_{d-1}}{p_0},\ldots,\frac{p_1}{p_0}\right).$$

Then we define one such hypercube centered in $\mathbf{x}$ having length of edge s by

$$W(\mathbf{x},s):=\left\{\mathbf{y}\in\mathbb{R}^{d-1}:\|\mathbf{x}-\mathbf{y}\|_\infty\leq\frac{s}{2}\right\}.$$

Finally we collect all those hypercubes that correspond to $\mathcal{C}_d(M)$.

$$\mathcal{W}_{d-1}(M):=\bigcup_{x\in\mathcal{C}_d(M)}W\left(\psi(\mathbf{x}),M^{-1}\right).$$

As indicated above we have to show, that the intersection of two such hypercubes is not too large. Let $\mathbf{x},\mathbf{y}\in\mathcal{C}_d(M)$ with $\mathbf{x}-\mathbf{y}=\mathbf{e}_j$ for some $j\in\{2,\ldots,d\}$. Then clearly

$$|\psi(\mathbf{x})_k-\psi(\mathbf{y})_k|=\begin{cases}0 & \text{if } k\neq d-j+1,\\ \frac{1}{M} & \text{if } k=d-j+1,\end{cases}$$

where $\psi(\mathbf{x})_k$ denotes the kth coordinate of the vector $\psi(\mathbf{x})$. Since for $\mathbf{u}\in W(\psi(\mathbf{x}),M^{-1})\cap W(\psi(\mathbf{y}),M^{-1})$ we have that $\|\psi(\mathbf{x})-u\|_\infty\leq(2M)^{-1}$ and $\|\psi(\mathbf{y})-u\|_\infty\leq(2M)^{-1}$ this implies that

$$\|\psi(\mathbf{x})-u\|_\infty=\|\psi(\mathbf{y})-u\|_\infty=\frac{1}{2M}.$$

Therefore $\mathbf{u}$ lies at the boarder of the hypercube and, since the boarder has Lebesgue measure zero, we get that

$$\lambda_{d-1}\left(W(\psi(\mathbf{x}),M^{-1})\cap W(\psi(\mathbf{y}),M^{-1})\right)=0. \tag{3.1}$$

Now we compare the number of elements in $\mathcal{C}_d(M)$ with the Lebesgue measure of $\mathcal{W}_{d-1}(M)$. We clearly have

$$\frac{|\mathcal{C}_d(M)|}{M^{d-1}}=\lambda_{d-1}\left(\mathcal{W}_{d-1}(M)\right).$$

The proof continues in two steps were we provide a lower and an upper bound for the number on the right side.

We start with the lower bound. To this end let $\mathbf{x}\in\mathcal{C}_d(M)$ such that $\psi(\mathbf{x})\in\mathcal{E}_{d-1}\left(1-c(2M)^{-\frac{1}{d-1}}\right)\subseteq\mathcal{D}_{d-1}$. For $\mathbf{y}\in W\left(\psi(\mathbf{x}),M^{-1}\right)$ we have that $\|\psi(\mathbf{x})-\mathbf{y}\|_\infty\leq\frac{1}{2M}$. Thus an application of

Lemma 2.1 implies $\rho(\mathbf{y}) < 1$ and therefore $\mathbf{y} \in \mathcal{D}_{d-1}$. Now we have that

$$\bigcup_{\substack{\mathbf{x}\in\mathcal{C}_d(M)\\ \rho(\psi(\mathbf{x}))<1-c(2M)^{-\frac{1}{d-1}}}} W\left(\psi(\mathbf{x}), M^{-1}\right) \subseteq \mathcal{D}_{d-1}. \tag{3.2}$$

Putting $\eta = c(2M)^{-\frac{1}{d-1}}$ together with an application of Lemma 2.2 yields

$$\lambda_{d-1}\left(\mathcal{D}_{d-1} \setminus \mathcal{E}_{d-1}(1-\eta)\right) \ll M^{-\frac{1}{d-1}}.$$

Now we concentrate on those polynomials $\mathbf{x} \in \mathcal{C}_d(M)$ whose spectral radius $\rho(\mathbf{x})$ is between $1-\eta$ and 1. To this end we define

$$\mathcal{L} := \{\mathbf{x} \in \mathcal{C}_d(M) | 1-\eta \leq \rho(\psi(\mathbf{x})) \leq 1\}.$$

Since the sets $\mathcal{D}_{d-1}$ and $\mathcal{E}_{d-1}(r)$ are defined by algebraic boundaries, the conditions of Lemma 2.3 are satisfied for $\mathcal{D}_{d-1} \setminus \mathcal{E}_{d-1}(1-\eta)$. Thus an application of Lemma 2.3 yields

$$\left| |\mathcal{L}| - M^{d-1}\lambda_{d-1}(\mathcal{D}_{d-1} \setminus \mathcal{E}_{d-1}(1-\eta)) \right| \ll M^{d-2}.$$

Combining this with (3.1) and (3.2) we obtain the lower bound

$$\lambda_{d-1}\left(\mathcal{D}_{d-1}\right) \geq \frac{|\mathcal{C}_d(M)|}{M^{d-1}}\left(1 - cM^{d-1-\frac{1}{d-1}}\right). \tag{3.3}$$

In order to provide an upper bound we construct an inverse function of ψ. In particular, for $M \in \mathbb{N}$ let $\chi_M : \mathbb{R}^{d-1} \to \mathbb{Z}^d$ be such that

$$\chi_M\left(r_{d-1}, \dots, r_1\right) = \left(M, \left\lfloor Mr_1 + \frac{1}{2}\right\rfloor, \dots, \left\lfloor Mr_{d-1} + \frac{1}{2}\right\rfloor\right).$$

Then we have for any $\mathbf{x} \in \mathcal{C}_d(M)$ that $\chi_M(\psi(\mathbf{x})) = \mathbf{x}$.

Now for an arbitrary $\mathbf{y} \in \mathcal{D}_{d-1}$ we set $\mathbf{x} := \chi_M(\mathbf{y})$. We clearly have that

$$\|\psi(\mathbf{x}) - \mathbf{y}\|_\infty \leq \frac{1}{2M}.$$

Thus an application of Lemma 2.1 yields

$$\rho(\psi(\mathbf{x})) \leq \rho(\mathbf{y}) + c(2M)^{-\frac{1}{d-1}} \leq 1 + c(2M)^{-\frac{1}{d-1}}.$$

Since $\mathbf{y} \in \mathcal{D}_{d-1}$ was arbitrary we get

$$\begin{aligned}\mathcal{D}_{d-1} &\subseteq \bigcup_{\substack{\mathbf{x}\in\mathbb{Z}^d\\ \rho(\psi(\mathbf{x}))<1+c(2M)^{-\frac{1}{d-1}}}} W\left(\psi(\mathbf{x}), M^{-1}\right)\\ &\subseteq \bigcup_{\mathbf{x}\in\mathcal{C}_d(M)} W\left(\psi(\mathbf{x}), M^{-1}\right) \cup \bigcup_{\substack{\mathbf{x}\in\mathbb{Z}^d\\ 1\leq\rho(\psi(\mathbf{x}))<1+c(2M)^{-\frac{1}{d-1}}}} W\left(\psi(\mathbf{x}), M^{-1}\right).\end{aligned}$$

Again we apply Lemma 2.2 with $\eta = c(2M)^{-\frac{1}{d-1}}$ and get

$$\lambda_{d-1}\left(\mathcal{E}_{d-1}(1+\eta) \setminus \mathcal{D}_{d-1}\right) \ll M^{-\frac{1}{d-1}}.$$

Since for $\mathcal{E}_{d-1}(1+\eta) \setminus \mathcal{D}_{d-1}$ the conditions of Lemma 2.3 are again satisfied we get that the number of $\mathbf{x} \in \mathbb{Z}^d$ such that $\psi(\mathbf{x})$ lies in $\mathcal{E}_{d-1}(1+\eta) \setminus \mathcal{D}_{d-1}$ is at most $\mathcal{O}\left(M^{d-1-\frac{1}{d-1}}\right)$. Thus

$$\lambda_{d-1}\left(\mathcal{D}_{d-1}\right) \leq \frac{|\mathcal{C}_d(M)|}{M^{d-1}}\left(1 + cM^{d-1-\frac{1}{d-1}}\right).$$

Together with the lower bound in (3.3) this proves the Theorem.

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(M. G. Madritsch) DEPARTMENT OF ANALYSIS AND COMPUTATIONAL NUMBER THEORY
GRAZ UNIVERSITY OF TECHNOLOGY
8010 GRAZ, AUSTRIA
E-mail address: `madritsch@math.tugraz.at`

(A. Pethő) DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF DEBRECEN,
NUMBER THEORY RESEARCH GROUP,
HUNGARIAN ACADEMY OF SCIENCES AND UNIVERSITY OF DEBRECEN
P.O. BOX 12, H-4010 DEBRECEN, HUNGARY
E-mail address: `petho.attila@inf.unideb.hu`