ON A SECOND CONJECTURE OF STOLARSKY: THE SUM OF DIGITS OF POLYNOMIAL VALUES

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Abstract. Let $q, r \geq 2$ be integers and denote by s_q the sum-of-digits function in base q. In 1978, K. B. Stolarsky conjectured that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \frac{s_2(n^r)}{s_2(n)} \leq r.$$

In this paper we prove this conjecture. We show that for polynomials $P_1(X), P_2(X) \in \mathbb{Z}[X]$ of degrees $r_1, r_2 \geq 1$ and integers $q_1, q_2 \geq 2$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))} = \frac{r_1(q_1-1)\log q_2}{r_2(q_2-1)\log q_1}.$$

We also present a variant of the problem to polynomial values of prime numbers.

1. Introduction and statement of results

Let $q \geq 2$ be an integer. Then every positive integer n has a unique q-adic representation of the form

$$n = \sum_{k=0}^{\ell} n_k q^k \quad \text{with} \quad n_\ell \neq 0.$$

We call a function f a q-additive function if it acts only on the digits of this expansion, *i.e.*,

$$f\left(\sum_{k=0}^{\ell} n_k q^k\right) = \sum_{k=0}^{\ell} f(n_k q^k).$$

Moreover, if this action is independent of the position of the digit, *i.e.*, $f(aq^k) = f(aq^j)$ for $k, j \geq 0$ and $a \in \{0, 1, \ldots, q-1\}$, then we call f strictly q-additive. The most famous example of a strictly q-additive function is the sum-of-digits function s_q defined by

$$s_q\left(\sum_{k=0}^{\ell} n_k q^k\right) = \sum_{k=0}^{\ell} n_k.$$

In 1978, K. B. Stolarsky [6] studied the distribution properties of the sequence of fractions

$$\left(s_2(n^r)/s_2(n)\right)_{n\geq 1},$$

Date: June 27, 2013.

2010 *Mathematics Subject Classification.* 11A63, 11N37.

Key words and phrases. sum-of-digits, mean value, polynomials, prime numbers.

The second author was supported by the Agence Nationale de la Recherche, grant ANR-10-BLANC 0103 MUNUM.

where $r \geq 2$ denotes a fixed integer. At the end of his paper, he posed two conjectures. His first conjecture was to give a proof that for all fixed $r \geq 2$ one has

$$\liminf_{n\to\infty} \frac{s_2(n^r)}{s_2(n)} = 0.$$

Hare, Laishram and Stoll [5] recently settled this conjecture and proved, more generally, that for any polynomial $P(X) \in \mathbb{Z}[X]$ with $P(\mathbb{N}) \subset \mathbb{N}$ of degree $r \geq 2$,

$$\liminf_{n\to\infty} \frac{s_q(P(n))}{s_q(n)} = 0.$$

Stolarsky also showed that the sequence $s_2(n^r)/s_2(n)$ is unbounded as $n \to \infty$ (this is also true for $\frac{s_q(P(n))}{s_q(n)}$, see [5]) and he posed the question whether

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \frac{s_2(n^r)}{s_2(n)} \tag{1.1}$$

exists and — if it exists — to determine its value. Stolarsky conjectured that the limit (1.1) exists and that it is included in the interval $]1, h]$. The purpose of this paper is to prove this conjecture. More precisely, as in Hare, Laishram and Stoll [5], we show a general version. We also present a variant to polynomial values of prime numbers. Let p_n denote the n–th prime, *i.e.*, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ etc.

Our main result is the following:

Theorem 1.1. *Let $q_1, q_2 \geq 2$ be integers and $P_1(X), P_2(X) \in \mathbb{Z}[X]$ be polynomials of degrees $r_1, r_2 \geq 1$, respectively, with $P_1(\mathbb{N}), P_2(\mathbb{N}) \subset \mathbb{N}$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))} = \frac{q_1 - 1}{q_2 - 1} \cdot \left(\frac{\log q_1}{\log q_2}\right)^{-1} \cdot \frac{r_1}{r_2}. \tag{1.2}$$

Moreover,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \frac{s_{q_1}(P_1(p_n))}{s_{q_2}(P_2(p_n))} = \frac{q_1 - 1}{q_2 - 1} \cdot \left(\frac{\log q_1}{\log q_2}\right)^{-1} \cdot \frac{r_1}{r_2}. \tag{1.3}$$

Remark 1.2. In exactly the same way one can show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \frac{s_{q_1}(P_1(p_n))}{s_{q_2}(P_2(n))} = \frac{q_1 - 1}{q_2 - 1} \cdot \left(\frac{\log q_1}{\log q_2}\right)^{-1} \cdot \frac{r_1}{r_2}.$$

All these results easily extend to strictly q_1- resp. q_2-additive functions provided that the variance and the image set of the functions satisfy some suitable conditions (see [1]). These conditions are automatically verified by the sum-of-digits function. We also remark that by using results due to Drmota and Steiner [3] one can prove analogous results for additive functions of polynomials in numeration systems that are defined via linear recurrent sequences, such as the Zeckendorf expansion.

2. Preliminaries

For the proof, we need some notation. We denote by

$$\mu_q = \frac{q-1}{2} \quad \text{and} \quad \sigma_q^2 = \frac{q^2-1}{12},$$

the mean and the variance of the values of the sum-of-digits function (see [1] or [2]). We will use the letter p to refer to a prime number, and use $\pi(N)$ for the number of primes up to N. We write $f \ll_\omega g$ or $f = O_\omega(g)$ if there exists a constant C depending at most on ω such that $f(x) \leq Cg(x)$ for sufficiently large x. If there is no such ω then the implied constant is meant to be absolute. We write $\log_q x$ for the logarithm to base q. Finally, for $A \subset \mathbb{N}$ we denote by $d(A)$ the asymptotic density of A, *i.e.*,

$$d(A) = \lim_{N\to\infty} \frac{A \cap [1,N]}{N}.$$

The idea of the proof of Theorem 1.1 is to use Cesàro means (see [4]). However, we cannot apply these means directly since the summands in (1.2) and (1.3) could be arbitrarily large. We will therefore divide the sequence into two parts. The first part corresponds to terms where the ratio stays close to the mean value whereas the second part is made up by terms that are far away from the mean (this will be made precise in a moment). For the first part, we use Cesàro means and the following

Lemma 2.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of reals and $A \subset \mathbb{N}$ a set with asymptotic density one. If*

$$\lim_{\substack{n\to\infty\\ n\in A}} x_n = x < \infty,$$

then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\leq N\\ n\in A}} x_n = x.$$

Proof. We define the sequence $(y_n)_{n\in\mathbb{N}}$ by

$$y_n = \begin{cases} x_n & \text{if } n \in A, \\ x & \text{if } n \notin A. \end{cases}$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\leq N\\ n\in A}} x_n = \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} y_n + \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\leq N\\ n\notin A}} x = x.$$

□

For the second part, we make use of a result of Bassily and Kátai [1].

Theorem 2.2. *[1, Theorem] Let $q \geq 2$ be an integer and $P(X) \in \mathbb{Z}[X]$ be a polynomial of degree $r \geq 1$ with $P(\mathbb{N}) \subset \mathbb{N}$. Then*

$$\frac{1}{N}\#\left\{1 \leq n \leq N \colon \frac{s_q(P(n)) - \mu_q \log_q(N^r)}{\sigma_q(\log_q N^r)^{\frac{1}{2}}} < t\right\} \xrightarrow[N\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x$$

and

$$\frac{1}{\pi(N)}\#\left\{1\le p\le N\colon \frac{s_q(P(p))-\mu_q\log_q(N^r)}{\sigma_q(\log_q N^r)^{\frac12}}<t\right\}\xrightarrow[N\to\infty]{}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x.$$

3. Proof of Theorem 1.1

In the sequel, assume that $N\ge 2$ is a fixed real number. For a given integer $q\ge 2$ and a given polynomial $P(X)\in\mathbb{Z}[X]$ (with $P(\mathbb{N})\subset\mathbb{N}$) of degree $r\ge 1$ we define $A_{q,P}=A_{q,P}(N)$ to be the set of integers n with $1\le n\le N$ such that $s_q(P(n))$ is close to its mean value, *i.e.*,

$$A_{q,P}:=\left\{1\le n\le N\colon \left|s_q(P(n))-\mu_q r\log_q N\right|\le\sigma_q(r\log_q N)^{\frac34}\right\}.$$

(We remark that in fact any exponent larger than $\frac12$ in place of $\frac34$ would have done the job.) In a similar way, we define $B_{q,P}=B_{q,P}(N)$ to be the set of integers n with $1\le n\le N$ such that $s_q(P(p_n))$ is close to its mean value (note that by the prime number theorem we have $p_N\sim N\log N$, as $N\to\infty$), *i.e.*,

$$B_{q,P}:=\left\{1\le n\le N\colon \left|s_q(P(p_n))-\mu_q r\log_q(N\log N)\right|\le\sigma_q(r\log_q(N\log N))^{\frac34}\right\}.$$

In order to be able to apply the properties of the Cesàro mean we need that both the numerator and the denominator of the ratios in (1.2) and (1.3) are near the mean. We first show that for $N\to\infty$ we have $B_{q,P}\sim N$ and $A_{q,P}\sim N$. We then use asymptotic densities to show that there are only few elements in $[1,N]\setminus(A_{q_1,P_1}\cap A_{q_2,P_2})$ resp. $[1,N]\setminus(B_{q_1,P_1}\cap B_{q_2,P_2})$. We will then be able to restrict our attention to $A_{q_1,P_1}\cap A_{q_2,P_2}$ resp. $B_{q_1,P_1}\cap B_{q_2,P_2}$ in the end.

We start with an application of Theorem 2.2. As $N\to\infty$, we get that

$$\begin{aligned}\#\left([1,N]\setminus A_{q,P}\right)&=\#\left\{1\le n\le N\colon \left|\frac{s_q(P(n))-\mu_q r\log_q N}{\sigma_q(r\log_q N)^{\frac12}}\right|>(r\log_q N)^{\frac14}\right\}\\&\ll N\int_{(r\log_q N)^{\frac14}}^{\infty}\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x.\end{aligned}$$

Thus the number of elements that lie not in $A_{q,P}$ can be estimated by the tail of the normal distribution. We have

$$\int_t^\infty\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x\le\int_t^\infty\frac{x}{t}\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x=\frac{\exp\left(-\frac{t^2}{2}\right)}{t},$$

where we have used that $0<t\le x$. Therefore,

$$\begin{aligned}\#\left([1,N]\setminus A_{q,P}\right)&\ll N\exp\left(-\frac{(r\log_q N)^{\frac12}}{2}\right)(r\log_q N)^{-\frac14}\\&\ll\frac{N}{(r\log_q N)^{\frac54}}\ll_{q,P}\frac{N}{(\log N)^{\frac54}}.\end{aligned}\tag{3.1}$$

The same calculation also shows that

(3.2)
$$\#\left([1,N]\setminus B_{q,P}\right) = \#\left\{1\le n\le N\colon \left|\frac{s_q(P(p_n))-\mu_q r\log_q(N\log N)}{\sigma_q(r\log_q(N\log N))^{\frac12}}\right| > (r\log_q(N\log N))^{\frac14}\right\}$$
$$\ll_{q,P} \frac{N}{(\log N)^{\frac54}}.$$

Recall the setting of Theorem 1.1. The prime number theorem (in the form $p_N\sim N\log N$) and a comparison of the lengths of the expansions give

$$\max\left(\frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))},\frac{s_{q_1}(P_1(p_n))}{s_{q_2}(P_2(p_n))}\right)\ll_{q_1,P_1}\ \log N,\qquad N\to\infty,$$

uniformly for all n with $1\le n\le N$. Hence, we get from (3.1) that

$$\frac1N\sum_{\substack{n\le N\\ n\notin A_{q_1,P_1}\cap A_{q_2,P_2}}}\frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))}\ll_{q_1,P_1}\frac{\log N}{N}\left(\sum_{\substack{n\le N\\ n\notin A_{q_1,P_1}}}1+\sum_{\substack{n\le N\\ n\notin A_{q_2,P_2}}}1\right) \tag{3.3}$$
$$\ll_{q_1,q_2,P_1}(\log N)(\log N)^{-\frac54}=o(1),$$

and similarly from (3.2) that

$$\frac1N\sum_{\substack{n\le N\\ n\notin B_{q_1,P_1}\cap B_{q_2,P_2}}}\frac{s_{q_1}(P_1(p_n))}{s_{q_2}(P_2(p_n))}=o(1). \tag{3.4}$$

Now we turn to the elements which are in $A_{q_1,P_1}\cap A_{q_2,P_2}$. By the definitions of these sets we get for all $n\in A_{q_1,P_1}\cap A_{q_2,P_2}$ (note that the denominator is positive),

$$\frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))}\le\frac{\mu_{q_1}r_1\log_{q_1}N+\sigma_{q_1}(r_1\log_{q_1}N)^{\frac34}}{\mu_{q_2}r_2\log_{q_2}N-\sigma_{q_2}(r_2\log_{q_2}N)^{\frac34}}$$
$$=\frac{\frac{\mu_{q_1}r_1}{\log q_1}+\sigma_{q_1}r_1^{\frac34}(\log_{q_1}N)^{-\frac14}(\log q_1)^{-\frac34}}{\frac{\mu_{q_2}r_2}{\log q_2}-\sigma_{q_2}r_2^{\frac34}(\log_{q_2}N)^{-\frac14}(\log q_2)^{-\frac34}},$$

and

$$\frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))}\ge\frac{\frac{\mu_{q_1}r_1}{\log q_1}-\sigma_{q_1}r_1^{\frac34}(\log_{q_1}N)^{-\frac14}(\log q_1)^{-\frac34}}{\frac{\mu_{q_2}r_2}{\log q_2}+\sigma_{q_2}r_2^{\frac34}(\log_{q_2}N)^{-\frac14}(\log q_2)^{-\frac34}}.$$

This can be rephrased as follows. Let $(N_k)_{k\ge0}$ be any sequence of reals with $\lim_{k\to\infty}N_k=\infty$ and let $(n_k)_{k\ge0}$ be any sequence of integers with $n_k\in A_{q_1,P_1}(N_k)\cap A_{q_2,P_2}(N_k)$. Then

$$\lim_{k\to\infty}\frac{s_{q_1}(P_1(n_k))}{s_{q_2}(P_2(n_k))}=\frac{\mu_{q_1}\log q_2}{\mu_{q_2}\log q_1}\cdot\frac{r_1}{r_2}.$$

A similar calculation shows that

$$\frac{s_{q_1}(P_1(p_n))}{s_{q_2}(P_2(p_n))} \leq \frac{\mu_{q_1} r_1 \log_{q_1}(N\log N) + \sigma_{q_1}\left(r_1 \log_{q_1}(N\log N)\right)^{\frac34}}{\mu_{q_2} r_2 \log_{q_2}(N\log N) - \sigma_{q_2}\left(r_2 \log_{q_2}(N\log N)\right)^{\frac34}}$$
$$= \frac{\frac{\mu_{q_1} r_1}{\log q_1}\left(1+\frac{\log\log N}{\log N}\right) + \sigma_{q_1} r_1^{\frac34}(\log q_1)^{-\frac34}(\log N)^{-\frac14}\left(1+\frac{\log\log N}{\log N}\right)^{\frac34}}{\frac{\mu_{q_2} r_2}{\log q_2}\left(1+\frac{\log\log N}{\log N}\right) - \sigma_{q_2} r_2^{\frac34}(\log q_2)^{-\frac34}(\log N)^{-\frac14}\left(1+\frac{\log\log N}{\log N}\right)^{\frac34}}.$$

In a similar way we get the lower bound with the signs reversed. Again, we obtain as limit

$$\lim_{k\to\infty} \frac{s_{q_1}(P_1(p_{n_k}))}{s_{q_2}(P_2(p_{n_k}))} = \frac{\mu_{q_1}\log q_2}{\mu_{q_2}\log q_1}\cdot\frac{r_1}{r_2}.$$

Now, since $A_{q,P}(N)/N \sim B_{q,P}(N)/N \sim 1$ as $N\to\infty$, the sets $\mathcal{A}_{q,P} = \bigcup_{N\geq1} A_{q,P}(N)$ and $\mathcal{B}_{q,P} = \bigcup_{N\geq1} B_{q,P}(N)$ satisfy $d(\mathcal{A}_{q,P}) = d(\mathcal{B}_{q,P}) = 1$, and therefore

$$d(\mathcal{A}_{q_1,P_1}\cap\mathcal{A}_{q_2,P_2}) = d(\mathcal{B}_{q_1,P_1}\cap\mathcal{B}_{q_2,P_2}) = 1.$$

By Lemma 2.1, the limit for $n\to\infty$ is not altered when we only look at those n that lie in these subsets of asymptotic density one. Thus we get

$$\lim_{N\to\infty}\frac1N \sum_{\substack{n\leq N\\ n\in\mathcal{A}_{q_1,P_1}\cap\mathcal{A}_{q_2,P_2}}} \frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))} = \frac{\mu_{q_1}\log q_2}{\mu_{q_2}\log q_1}\cdot\frac{r_1}{r_2}. \tag{3.5}$$

A combination of (3.3) and (3.5) yields

$$\begin{aligned}\lim_{N\to\infty}\frac1N\sum_{n\leq N}&\frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))}\\ &= \lim_{N\to\infty}\frac1N \sum_{\substack{n\leq N\\ n\in\mathcal{A}_{q_1,P_1}\cap\mathcal{A}_{q_2,P_2}}} \frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))} + \lim_{N\to\infty}\frac1N \sum_{\substack{n\leq N\\ n\notin\mathcal{A}_{q_1,P_1}\cap\mathcal{A}_{q_2,P_2}}} \frac{s_{q_1}(P_1(n))}{s_{q_2}(P_2(n))}\\ &= \frac{\mu_{q_1}\log q_2}{\mu_{q_2}\log q_1}\cdot\frac{r_1}{r_2},\end{aligned}$$

which proves (1.2), and similarly we get (1.3). This completes the proof of Theorem 1.1. □

4. Concluding remarks

The above results do not hold in general for arbitrary q-additive functions. For example, let $f\colon\mathbb{N}\to\mathbb{N}$ be such that $f(n)=1$ for $n\geq 0$. This function is q-additive (for any $q\geq2$) as it puts 1 on the least significant digit of n and 0 on all other digits. Then, for each $r\geq1$, we have that

$$\lim_{N\to\infty}\frac1N\sum_{n=1}^N \frac{f(n^r)}{f(n)} = 1.$$

On the other hand, if $f\colon\mathbb{N}\to\mathbb{N}$ is q-additive with $f(aq^k) = 2^k$ for $a\in\{0,\dots,q-1\}$ and $k\geq0$ then for each $r\geq2$,

$$\lim_{N\to\infty}\frac1N\sum_{n=1}^N \frac{f(n^r)}{f(n)} = \infty.$$

In order to get other non-trivial values for the limit, the values of f must depend on the position k as well as on the digit a. We conclude our discussion with the following conjecture.

Conjecture. *Let $q, r \geq 2$ be integers. Then for each real $\ell \in [1, \infty)$ there exists a q-additive function $f\colon \mathbb{N} \to \mathbb{N}$ such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{f(n^r)}{f(n)} = \ell.$$

Acknowledgment

The authors want to thank Peter J. Grabner (Graz University of Technology) and Kevin G. Hare (University of Waterloo) for fruitful discussions on this problem.

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